HOCHSCHILD COHOMOLOGY AND STRING TOPOLOGY FOR ORBIFOLDS

ANDRÉS ÁNGEL, ERIK BACKELIN AND BERNARDO URIBE

Abstract. Let M be a connected, simply connected, closed and oriented manifold, and G a finite group acting on M by orientation preserving diffeomorphisms. In this paper we show an explicit ring isomorphism between the orbifold string topology of the orbifold $[M/G]$ and the Hochschild cohomology of the dg-ring obtained by performing the smash product between the group G and the singular cochain complex of M.

1. Introduction

String topology stands for the study of the topological properties associated to the space of smooth free loops $\mathcal{L}M$ on a closed and oriented manifold M of dimension d.

The starting point of string topology was the paper [3] by Chas and Sullivan where the authors discovered an intersection product in the homology of the free loop space

$$H_p(\mathcal{L}M) \otimes H_q(\mathcal{L}M) \to H_{p+q-d}(\mathcal{L}M),$$

having total degree $-d$, which together with the degree 1 operator $H_*(\mathcal{L}M) \to H_{*+1}(\mathcal{L}M)$ induced by the circle action on the loops, endowed the homology of the free loop space with the structure of a Batalin-Vilkovisky algebra.

Cohen and Jones [5] developed a homotopical theoretic realization of string topology, by endowing the Thom spectrum $\mathcal{L}M^{-TM}$ with the structure of a ring spectrum

$$\mathcal{L}M^{-TM} \wedge \mathcal{L}M^{-TM} \to \mathcal{L}M^{-TM},$$

that allowed them to show, that at the level of homology, the intersection product of Chas and Sullivan was recovered by the product in homology induced by the ring spectrum $\mathcal{L}M^{-TM}$. In the same paper, Cohen and Jones furthermore showed, generalizing results of Jones [14], that in the case when M is simply connected, there is a ring isomorphism between the

2010 *Mathematics Subject Classification.* 55P50, 57R18 (primary), 18G55, 18E30 (secondary).

Key words and phrases. Orbifolds, string topology, Hochschild cohomology.

The third author akcnowledges support provided by COLCIENCIAS through grant number 120440520139. The third author was partially supported with funds from the Leibniz prize of Prof. Dr. Wolfgang Lück .

homology of the ring spectrum $\mathcal{L}M^{-TM}$ and the Hochschild cohomology of the singular cochains of M

$$H_*(\mathcal{L}M^{-TM}) \cong HH^*(C^*M, C^*M).$$

In the case of global quotient orbifolds of the form $[M/G]$ for G finite group, Lupercio and the third author [17] constructed the loop groupoid $[P_GM/G]$ as the natural free loop space of the orbifold, whose homotopic quotient turns out to be homotopy equivalent to the space of free loops of the homotopy quotient $M \times_G EG$, i.e.

$$P_GM \times_G EG \simeq \mathcal{L}(M \times_G EG);$$

and with this equivalence in hand, Lupercio, Xicoténcatl and the third author [18] showed that the homology of the free loop space of the orbifold

$$H_*(\mathcal{L}(M \times_G EG); \mathbb{Q}) \cong H_*(P_GM; \mathbb{Q})^G$$

could also be endowed with the structure of a Batalin-Vilkovisky algebra; the authors coined this structure with the name *orbifold string topology.*

But, can the orbifold string topology be defined over the homology with integer coefficients? And, is there any relation between the orbifold string topology ring and the Hochschild cohomology of some specific dg-ring? This paper is devoted to positively answer these two questions.

Let us start with a brief description of the answer of the second question, as its solution leads the way to solve the first. From the isomorphism showed by Cohen and Jones between the string topology ring of a manifold and the Hochschild cohomology of the singular cochains, one is tempted to try to show that the orbifold string ring should be isomorphic to the ring

$$HH^*(C^*(M \times_G EG), C^*(M \times_G EG));$$

but due to convergence issues (the Eilenberg-Moore spectral sequence does not converge in general [6]), one cannot show using standard cosimplicial methods that indeed this ring recovers the orbifold string ring, and even worse, in some cases we show (see section 8) that this ring does not give the appropriate ring structure on the homology of the free loop space of $[M/G]$. Instead we consider the dg-ring $C^*M\#G$ defined as the smash product of G with the singular cochains $C^*M = C^*(M; \mathbb{Z})$ of M, and we compare its Hochschild cohomology with the homology ring of the ring spectrum P_GM^{-TM} that was constructed in [18].

In the case that M is simply connected and connected we find that that the orbifold string topology ring can be recovered as the Hochschild cohomology ring of $C^*(M; \mathbb{Q})\#G$, i.e. there is an isomorphism of rings

$$HH^*(C^*(M; \mathbb{Q})\#G, C^*(M; \mathbb{Q})\#G) \cong H_*(P_GM^{-TM}; \mathbb{Q})^G.$$

This isomorphism is obtained by carefully decomposing the Hochschild cohomology ring into smaller parts, which leads to the ring isomorphism

$$HH^*(C^*M\#G, C^*M; \#G) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, C_*(P_GM^{-TM})).$$

With the previous isomorphism in hand, it iwas clear that in order to get a topological counterpart to the Hochschild cohomology of $C^*M\#G$, it was necessary to introduce some sort of Poincaré dual to the universal principal G bundle EG. Fortunately the spaces EG can be approximated by finite dimensional manifolds EG_n with free G actions, which together with the S-duality identification

$$C^*(EG_n) \simeq C_*(EG_n^{-TEG_n})$$

allow us to construct a pro-ring spectrum whose homology

$$H_*^{pro}(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}) := \lim_{\leftarrow n} H_*((P_G M \times_G EG_n)^{-e_0^* T(M\times_G EG_n)})$$

turned out to be isomorphic to the Hochschild cohomology of $C^*M\#G$

$$HH^*(C^*M\#G, C^*M\#G) \cong H_*^{\mathrm{pro}}\left(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}\right).$$

This is the main theorem of the paper (theorem 6.3).

Because of this last isomorphisms we define the orbifold string topology ring with integer coefficients to be the ring

$$H_*^{\mathrm{pro}}\left(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}\right),$$

and in this way we answer also the first question.

The main effort of this paper is to prove the main theorem (theorem 6.3) and for that purpose we have devoted the first 5 sections. After that follows some simple applications of the main theorem and in the last section 8 we explain why Hochschild cohomology is not preserved under groupoid equivalence.

When we started this project we found that the literature about Hochschild cohomology of dg-rings was a little disperse, it tended to either be based on formulas whose categorical meaning were not properly explained or gave much more sophisticated expositions than was required for our purposes. Therefore we decided to give a detailed and elementary description of the homological aspects of Hochschild cohomology that both describe the concrete formulas needed to compute things as well as the interpretation of Hochschild cohomology as Ext-groups in the derived category of dg-modules. This also clarifies the relationships between algebraic constructions associated to $C^*M\#G$ and topological constructions that deal with free loop spaces.

The layout of the paper is as follows. In section 2 we give the preliminaries on derived categories of dg-modules over dg-rings and we describe several equivalent ways on which the Hochschild cohomology of a dg-ring $\mathcal{A}$ can be defined. In section 3 we consider the smash product of a discrete group G with a dg-ring $\mathcal{A}$ and we give an explicit resolution of $\mathcal{A}\#G$ as a $\mathcal{A}\#G^e$-module, that allows us to decompose the complex that produces the

Hochschild cohomology of $\mathcal{A}\#G$ into the composition of two functors

$$\mathcal{RH}om_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{RH}om_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)).$$

In section 4 we construct cosimplicial spaces $\mathbb{P}_g M$ whose total spaces realize the orbifold loops $P_g M$, and with this identification in hand we construct a quasi-isomorphism

$$C^*M \overset{L}{\otimes}_{C^*M^e} C^*M\#G \simeq C^*(P_G M).$$

In section 5 we construct cosimplicial spectra $\mathfrak{P}_g M$ whose total spectra realize the spectra $P_g M^{-TM}$, and that moreover permits to construct a quasi-isomorphism

$$\mathcal{H}om_{C^*M^e}(B(C^*M), C^*M\#G) \simeq C_*(P_G M^{-TM})$$

where $B(C^*(M))$ denotes the Bar construction of $C^*(M)$. And in section 6 we prove the main theorem of the paper (theorem 6.3) by constructing a pro ring spectrum for the orbifold $[M/G]$ whose homology ring is isomorphic to the Hochschild cohomology of $C^*M\#G$.

In section 7 we show some applications of the main theorem, and in section 8 we show why the Hochschild cohomology is not an invariant of the orbifold by presenting two equivalent groupoids with different Hochschild cohomologies. We finish with two Appendices, the first in section 9 is devoted to explain the sign notations that are used in the Bar construction, and the second in 10, is devoted to show that all the constructions performed in this paper for dg-rings that are free with respect to $\mathbb{Z}$, can be applied to the dg-ring of singular cochains $C^*(M)$ on a manifold which in general is not free over $\mathbb{Z}$. This second Appendix puts in solid grounds the results of sections 5 and 6.

Acknowledgments. The authors thank the hospitality of the Max Planck Institut für Mathematik in Bonn, where part of this paper was carried out. The third author would like to thank Prof. Dr. Wolfgang Lück for his support at the final stages of this project.

2. Derived category of dg-modules over a dg-ring

In this section we will give the preliminaries on differential graded modules over differential graded rings and we will setup the notation. We give a rather detailed exposition partly because we have felt there was a need in the literature for an elementary introduction to Hochschild (co)homology in the dg-setting that relates the derived category of dg-modules and the derived functor approach to the down to earth formulas used by the working topologists and algebraists.

This summary is based on the papers [24, 15] and the book [1]. In what follows all complexes will be cohomological i.e. the differentials will raise the degree by one.

A **differential graded ring** (dg-ring) is a pair $\mathcal{A} = (A, d)$ consisting of a $\mathbb{Z}$-graded ring A together with a differential d of degree 1 which satisfies the Leibniz rules

$$d(ab) = d(a)b + (-1)^{|a|} ad(b)$$

for all homogeneous elements $a, b \in A$. In this paper we will assume that the dg-rings come endowed with a unit element. A is called (graded) commutative if $ab = (-1)^{|a||b|} ba$. For the rest of this section $\mathcal{A}$ denotes a dg-ring.

A **differential graded left $\mathcal{A}$-module** (left $\mathcal{A}$-module) consists of a graded left A-module M together with a differential d_M of degree 1 which satisfies the Leibniz rule

$$d_M(ab) = d(a)b + (-1)^{|a|} ad_M(b)$$

for all homogenous elements $a \in A$ and $b \in M$. A morphism $f : M \to N$ of $\mathcal{A}$-modules is an A-linear map which is homogenous of degree 0 and commutes with the differentials.

Denote by $\mathcal{A} - \text{mod}$ the category of left $\mathcal{A}$-modules. We denote homomorphisms in this category by

$$\mathrm{Hom}_{\mathcal{A}}(\_\,,\_\,) := \mathrm{Hom}_{\mathcal{A}-\mathrm{mod}}(\_\,,\_\,).$$

The category of graded left A-modules, is denoted by $A - \text{mod}$. Morphisms in this category are by definition A-linear maps homogenous of degree 0. We put

$$\mathrm{Hom}_{A}(\_\,,\_\,) := \mathrm{Hom}_{A-\mathrm{mod}}(\_\,,\_\,).$$

Thus

$$\mathrm{Hom}_{\mathcal{A}}(M, N) = \{f \in \mathrm{Hom}_A(M, N) | d_N \circ f - f \circ d_M = 0\}.$$

Similarly, there are the categories $\text{mod} - \mathcal{A}$ of right $\mathcal{A}$-modules and $\mathcal{A} - \text{mod} - \mathcal{A}$ of $\mathcal{A}$-bimodules.

The **opposite** dg-ring $\mathcal{A}^o$ of $\mathcal{A}$ is defined to be $\mathcal{A}^o = (A^o, d)$ where its elements are the same ones as in $\mathcal{A}$ and with the same differential but the multiplication $a \circ b$ is the opposite of the one in A, i.e.

$$a \circ b := (-1)^{|a||b|} ba.$$

Note that if $\mathcal{A}'$ is any dg-ring then $\mathcal{A} \otimes_{\mathbb{Z}} \mathcal{A}'$ is a dg-ring with multiplication $a \otimes a' \cdot b \otimes b' = (-1)^{|a'||b|} ab \otimes a'b'$ and differential $d(a \otimes a') = da \otimes a' + (-1)^{|a'|} a \otimes da'$, for $a, b \in \mathcal{A}$ and $a', b' \in \mathcal{A}'$.

The **shift functor** [1] on $\mathcal{A} - \text{mod}$ is given by shifting the degree of a complex by one

$$M[1]^k = M^{1+k}$$

and with differential $d_{M[1]} = -d_M$. We therefore have a canonical isomorphism of degree 1

$$s : M \to M[1] \quad \text{such that} \qquad sd_M x = -d_{M[1]} sx. \tag{2.1}$$

The **tensor product** of a right $\mathcal{A}$-module M and a left $\mathcal{A}$-module N consists of the graded complex of abelian groups $M \otimes_A N$ together with the differential

$$d(m \otimes n) = d_M m \otimes n + (-1)^{|m|} m \otimes d_N n.$$

If A is graded commutative then M is automatically an $\mathcal{A}$-bimodule by $amb = (-1)^{|m||b|} mab$, for $a, b \in A$ and $m \in M$. Hence, in this case $M \otimes_{\mathcal{A}} N$ is a left $\mathcal{A}$-module by $a \cdot m \otimes n = (am) \otimes n$.

A homomorphism $f : M \to N$ in $\mathcal{A} - \text{mod}$ is a **quasi-isomorphism** if f induces an isomorphism in cohomology $H^* f : H^* M \cong H^* N$.

A **chain homotopy** between homomorphisms $f_0, f_1 \in \text{Hom}_{\mathcal{A}}(M, N)$ of $\mathcal{A}$-modules is a homomorphism of graded A-modules $k \in \text{Hom}_A(M, N[-1])$ of degree -1 which satisfies

$$f_1 - f_0 = d_M \circ k + k \circ d_N.$$

The **homotopy category** $\mathcal{K}(\mathcal{A})$ of the dg-ring $\mathcal{A}$ has the same objects as $\mathcal{A} - \text{mod}$ and as morphisms the chain homotopy equivalence classes of morphisms in $\mathcal{A} - \text{mod}$.

The **derived category** $\mathcal{D}(\mathcal{A})$ is the localization of the homotopy category $\mathcal{K}(\mathcal{A})$ with respect to quasi-isomorphisms.

In more sophisticated presentations than ours one define a structure of Quillen model category on $\mathcal{A} - \text{mod}$, see [13, 22]. Although this topic is not discussed here, we use below the model category term "cofibrant" for objects which essentially do the same work in the dg-world (where objects, e.g., the bar construction, automatically behave like unbounded complexes) as bounded above projective resolutions do in classical homological algebra.

A (left $\mathcal{A}$-)module M is **cofibrant with respect to** $\mathcal{A}$ if there exists an exhaustive increasing filtration by submodules

$$0 = M^0 \subset M^1 \subset M^2 \subset \cdots \subset M^k \subset \cdots \subset M$$

such that each subquotient M^k/M^{k-1} is a direct summand of a direct sum of shifted copies of $\mathcal{A}$. An $\mathcal{A}$-module M is **cofibrant** if it is cofibrant with respect to both $\mathcal{A}$ and the base ring $\mathbb{Z}$. In the case that $\mathcal{A}$ were free over $\mathbb{Z}$, the cofibrant modules are the same as the cofibrant modules with respect to $\mathcal{A}$.

Up to chain homotopy equivalence, the cofibrant modules are the ones that possesses Keller's property (P) [15]. Every quasi-isomorphism between cofibrant modules is a chain homotopy equivalence and every module can be approximated up to quasi-isomorphism by a cofibrant module.

The derived category $\mathcal{D}(\mathcal{A})$ is equivalent to the full subcategory of $\mathcal{K}(\mathcal{A})$ whose objects are cofibrant $\mathcal{A}$-modules, see [15].

If $f : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of dg-rings, then the derived functors of restriction and extension of scalars induce equivalences between the derived categories $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$.

Let M and N be left $\mathcal{A}$-modules, the **homomorphism complex**

$$\mathcal{H}om_{\mathcal{A}}(M, N)$$

between M and N is the complex defined as follows: in dimension $k \in \mathbb{Z}$ the chain group $\mathcal{H}om_{\mathcal{A}}(M,N)^k$ is the group of graded A-modules homomorphisms of degree k, i.e.

$$\mathcal{H}om_{\mathcal{A}}(M,N)^k := \mathrm{Hom}_A(M, N[k]),$$

and the differential

$$d : \mathcal{H}om_{\mathcal{A}}(M,N)^k \to \mathcal{H}om_{\mathcal{A}}(M,N)^{k+1}$$

is defined by

$$d(f) := d_N \circ f - (-1)^k f \circ d_M.$$

With this definition in mind, the 0-cycles of the homomorphism complex $\mathcal{H}om_{\mathcal{A}}(M,N)$ are precisely the $\mathcal{A}$-module homomorphisms between M and N:

$$\mathcal{Z}^0\mathcal{H}om_{\mathcal{A}}(M,N) = \mathrm{Hom}_{\mathcal{A}}(M,N)$$

and the zero-th cohomology of the complex $\mathcal{H}om_{\mathcal{A}}(M,N)$ is precisely the set of equivalence classes of chain homotopic maps:

$$H^0\mathcal{H}om_{\mathcal{A}}(M,N) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(M,N).$$

For M, N and L left $\mathcal{A}$-modules composition of maps induces a bilinear pairing

$$\mathcal{H}om_{\mathcal{A}}(N,L)^k \times \mathcal{H}om_{\mathcal{A}}(M,N)^l \to \mathcal{H}om_{\mathcal{A}}(M,L)^{k+l} \tag{2.2}$$

that moreover satisfies the Leibniz rule: for $f \in \mathcal{H}om_{\mathcal{A}}(M,N)^l$ and $g \in \mathcal{H}om_{\mathcal{A}}(N,L)^k$ one can check that

$$d(g \circ f) = dg \circ f + (-1)^k g \circ df.$$

From this it follows in particular that the endomorphism complex $\mathcal{H}om_{\mathcal{A}}(M,M)$ is a differential graded ring with multiplication the composition of homomorphisms.

2.1. **Derived functors.** For M a left $\mathcal{A}$-module and N a right $\mathcal{A}$-module, the derived tensor $\overset{L}{\otimes}_{\mathcal{A}}$ between N and M is defined as the complex

$$N \overset{L}{\otimes}_{\mathcal{A}} M := N \otimes_{\mathcal{A}} M'$$

where $M' \to M$ is a cofibrant replacement for M. The derived tensor product defines a functor $\overset{L}{\otimes}_{\mathcal{A}}$: $\mathcal{D}(\mathcal{A}^o) \times \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathbb{Z})$.

The Tor-groups between N and M are defined as the cohomology of the derived tensor between N and M,

$$\mathrm{Tor}^*_{\mathcal{A}}(N,M) = H^*(N \overset{L}{\otimes}_{\mathcal{A}} M).$$

Let M and N be two $\mathcal{A}$-modules, the derived functor of $\mathcal{H}om_{\mathcal{A}}$ is the functor $\mathcal{RH}om_{\mathcal{A}}$ which is defined as the complex

$$\mathcal{RH}om_{\mathcal{A}}(M,N) := \mathcal{H}om_{\mathcal{A}}(M',N)$$

where $M' \to M$ is a cofibrant replacement of M. We see that $\mathcal{RH}om$ is well defined up to chain homotopies and it defines a functor $\mathcal{RH}om_{\mathcal{A}}(\ ,\ ) : \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathbb{Z})$.

By definition we have

$$\mathrm{Ext}^k_{\mathcal{A}}(M,N) := \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(M, N[k]) = H^k \mathcal{RH}om_{\mathcal{A}}(M,N) =$$

$$H^k \mathcal{H}om_{\mathcal{A}}(M', N).$$

We have seen that the endomorphism complex

$$\mathcal{H}om_{\mathcal{A}}(M', M')$$

becomes a differential graded ring by composition of homomorphisms. This in particular implies that the graded Ext-group

$$\mathrm{Ext}^*_{\mathcal{A}}(M,M) = H^* \mathcal{H}om_{\mathcal{A}}(M', M') \tag{2.3}$$

becomes a graded ring.

2.2. **Hochschild (co)homology for dg-rings.** In this section we define the Hochschild homology and cohomology for a dg-ring $\mathcal{A}$ and we will list some of its properties.

Consider the dg-ring

$$\mathcal{A}^e := \mathcal{A} \otimes_{\mathbb{Z}} \mathcal{A}^o.$$

Then an $\mathcal{A}$-bimodule is the same thing as a left $\mathcal{A}^e$-module. Let us consider $\mathcal{A}$ as a left $\mathcal{A}^e$-module in the natural way.

In order to define the Hochschild (co)homology we need to define a replacement of $\mathcal{A}$ in $\mathcal{A}^e - \mathrm{mod}$ that it is known as the Bar construction.

2.2.1. *Bar construction.* The Bar construction is based on the Bar resolution for modules over rings. We will use the sign conventions defined in [9, 8] and in section 9 we will show how these sign conventions arise.

For $k \geq 0$ let

$$P^{-k} := \left(\mathcal{A} \otimes \mathcal{A}[1]^{\otimes k} \otimes \mathcal{A}\right) = \mathcal{A}^{\otimes k+2}[k]$$

be the $\mathcal{A}^e$-module defined in the natural way by

$$(a \otimes b)(x_0|x_1|\dots|x_{k+1}) = (ax_0|x_1|\dots|x_{k+1}b)$$

where $a \otimes b \in \mathcal{A}^e$, and $(x_0|x_1|\dots|x_{k+1})$ denotes the element

$$x_0 \otimes sx_1 \otimes \cdots \otimes sx_n \otimes x_{j+1}$$

in P^{-k} where sx denotes the image in $\mathcal{A}[1]$ of $x \in \mathcal{A}$ under the isomorphism $s : \mathcal{A} \to \mathcal{A}[1]$ induced by the shift functor; see (2.1).

We have that the degree of an element in P^{-k} is

$$|(a_0|\dots|a_{k+1})| := |a_0| + \cdots + |a_{k+1}| - k.$$

and the differential in P^{-k} becomes

$$\begin{aligned} d(x_0|x_1|\dots|x_{k+1}) \quad &:= \quad (dx_0|x_1|\dots|x_{k+1}) - \sum_{j=1}^{k}(-1)^{\varepsilon_{j-1}}(x_0|\dots|dx_j|\dots|x_{k+1}) \\ &\qquad +(-1)^{\varepsilon_k}(x_0|\dots|x_k|dx_{k+1}) \end{aligned} \tag{2.4}$$

where

$$\varepsilon_j := |x_0| + |x_1| + \cdots + |x_j| - j$$

denotes the degree of the first $j+1$ elements of $(a_0|\dots|a_{k+1})$ as an element in $\mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A}$.

Define the homomorphisms of $\mathcal{A}^e$-modules

$$\begin{aligned} \delta^{-k} : P^{-k} \quad &\to \quad P^{-k+1} \\ (x_0|x_1|\dots|x_{k+1}) \quad &\mapsto \quad \sum_{j=0}^{k-1}(-1)^{\varepsilon_j}(x_0|\dots|x_{j-1}|x_jx_{j+1}|x_{j+2}|\dots|x_{k+1}) \\ &\qquad -(-1)^{\varepsilon_k}(x_0|\dots|x_{k-1}|x_kx_{k+1}) \end{aligned}$$

which together with the module homomorphism

$$\begin{aligned} \epsilon : P^0 \quad &\to \quad \mathcal{A} \\ (x_0|x_1) \quad &\mapsto \quad x_0x_1 \end{aligned}$$

determines a complex of $\mathcal{A}^e$-modules over

$$\cdots P^{-3} \xrightarrow{\delta^{-3}} P^{-2} \xrightarrow{\delta^{-2}} P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\epsilon} \mathcal{A} \longrightarrow 0$$

which turns out to be acyclic if we consider it as a complex of modules over A^e.

The bar construction is the $\mathcal{A}^e$-module

$$B(\mathcal{A}) := \bigoplus_{k=0}^{\infty} P^{-k} = \bigoplus_{k=0}^{\infty} \Big(\mathcal{A} \otimes \mathcal{A}[1]^{\otimes k} \otimes \mathcal{A}\Big)$$

with differential

$$\begin{aligned} D : P^{-k} \quad &\to \quad P^{-k} \oplus P^{-k+1} \\ D(p) \quad &= \quad d_{P^{-k}}p + \delta_{-k}p \end{aligned} \tag{2.5}$$

that we will simply denote by $D = d + \delta$.

It is straightforward to check that the differentials d and δ commute as operators, i.e. $[d,\delta] = d\delta - (-1)^{|d||\delta|}\delta d = 0$, and therefore we have that $D^2 = 0$.

The sign conventions for the differentials d and δ are obtained by transporting the structure of the usual differentials d and the Bar differential δ from $\oplus_k \mathcal{A}^{k+2}$ to $\oplus_k \left(\mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A}\right)$. See section 9 for a proof of this fact.

We extend the map ϵ of (2.5) to a morphism with $\epsilon : B(\mathcal{A}) \to \mathcal{A}$ (the augmentation morphism) by requiring $\epsilon|_{P^{-k}} = 0$ for $k > 0$.

We include a proof of the following well known result.

Lemma 2.1. *Assume that the dg-ring $\mathcal{A}$ is free over $\mathbb{Z}$. Then $B(\mathcal{A}) \xrightarrow{\epsilon} \mathcal{A}$ is a cofibrant replacement of $\mathcal{A}$.*

Proof. Let us consider the filtration of $B(\mathcal{A})$ defined by

$$F^q = \{(x_0|\dots|x_{k+1}) \in B(\mathcal{A}) \mid q \leq |x_0| + |x_1| + \cdots |x_{k+1}|\}.$$

The E^0 term of the spectral sequence associated to the filtration F^* is isomorphic to the bar construction $B(A)$ of the graded algebra A. Therefore the E^1 term of the spectral sequence is isomorphic to A with differential the same differential of $\mathcal{A}$. The spectral sequence collapses at level 2 and the term E^2 is therefore isomorphic to $H^*\mathcal{A}$. We conclude from this that the map ϵ induces an isomorphism in cohomologies $\epsilon : H^*B(\mathcal{A}) \cong H^*\mathcal{A}$.

To prove that $B(\mathcal{A})$ is cofibrant we use the filtration defined as follows: for $q \geq 0$

$$F^{2q} := \bigoplus_{k=0}^{q} \Big(\mathcal{A} \otimes \mathcal{A}^{\otimes k} \otimes \mathcal{A}\Big)[k]$$

and

$$F^{2q+1} := F^{2q} \oplus \big(\mathcal{A} \otimes W^{q+1} \otimes \mathcal{A}\big)[q+1]$$

where W^q is the $\mathcal{A}^e$-submodule of $\mathcal{A}^{\otimes q}$ defined as the kernel of the differential d, i.e.

$$W^q := \mathrm{Ker}(d : \ \mathcal{A}^{\otimes q} \to \mathcal{A}^{\otimes q}).$$

Is is clear that

$$\phi \subset F^0 \subset F^1 \subset F^2 \subset \cdots B(\mathcal{A}),$$

that $\cup_k F^k = B(\mathcal{A})$ and that F^k is a submodule of $B(\mathcal{A})$. The subquotients of the filtration are isomorphic to the $\mathcal{A}^e$-modules

$$F^{2q+1}/F^{2q} \cong \big(\mathcal{A} \otimes W^{q+1} \otimes \mathcal{A}\big)[q+1]$$

$$F^{2q}/F^{2q-1} \cong \big(\mathcal{A} \otimes (\mathcal{A}^{\otimes q}/W^q) \otimes \mathcal{A}\big)[q]$$

where in both cases the induced differential is zero in the middle part: the differential is zero on W^{q+1} by definition, and as the image of $d : \mathcal{A}^{\otimes q} \to \mathcal{A}^{\otimes q}$ is a subset of W^q, then the differential on $\mathcal{A}^{\otimes q}/W^q$ is also zero.

Note that W^{q+1} and $\mathcal{A}^{\otimes q}/W^q \cong \mathrm{Im}(d : \mathcal{A}^{\otimes q} \to \mathcal{A}^{\otimes q})$ are free over $\mathbb{Z}$, because both of them are $\mathbb{Z}$-submodules of $\mathcal{A}^{\otimes q}$ which is free over $\mathbb{Z}$. Therefore $F^{2q+1}/F^{2q}[-q-1]$ and $F^{2q}/F^{2q-1}[-q]$, respectively, are free left $\mathcal{A}^e$-modules, with $\mathcal{A}^e$-basis $\{1 \otimes x \otimes 1\}_x$, as x runs over a $\mathbb{Z}$-module basis for W^{q+1} and $\mathcal{A}^{\otimes q}/W^q$, respectively. Thus the complex $B(\mathcal{A})$ is cofibrant. □

2.2.2. *Definition of Hochschild (co)homology.* With the bar construction in hand we can define the Hochschild (co)homology groups.

Definition 2.2. The Hochschild cohomology ring of a dg-ring $\mathcal{A}$ is the ring

$$HH^*(\mathcal{A}, \mathcal{A}) := H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A}))$$

where the ring structure is given by composition of maps, and the Hochschild homology groups is given by the graded group

$$HH_*(\mathcal{A},\mathcal{A}) := H^*(B(\mathcal{A})\otimes_{\mathcal{A}^e}\mathcal{A}).$$

In the case that $\mathcal{A}$ is free over $\mathbb{Z}$, we know from lemma 2.1 that $B(\mathcal{A})$ is a cofibrant replacement for $\mathcal{A}$ in $\mathcal{A}^e$-mod, and therefore we could alternatively define the Hochschild cohomology as the graded ring

$$HH^*(\mathcal{A},\mathcal{A}) := \mathrm{Ext}^*_{\mathcal{A}^e}(\mathcal{A},\mathcal{A}).$$

and the Hochschild homology as the graded group

$$HH_*(\mathcal{A},\mathcal{A}) := \mathrm{Tor}^*_{\mathcal{A}^e}(\mathcal{A},\mathcal{A}).$$

2.2.3. *Properties of the Hochschild cohomology.* The Hochschild cohomology can be also calculated using the complex

$$\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}).$$

This complex is better suited for the topological constructions that are done in the rest of the paper in order to relate the homology of the free loops on a manifold and the Hochschild cohomology of the dg-ring of cochains.

We define the product $\phi\cdot\psi$ of $\phi,\psi\in\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ by

$$(\phi\cdot\psi)(a_0|\dots|a_{k+1}) \quad = \quad \sum_{j=0}^{k}(-1)^{|\psi|\varepsilon_j}\phi(a_0|\dots|a_j|1)\psi(1|a_{j+1}|\dots|a_{k+1}).$$

We shall show that this ring structure makes $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ into a dg-ring, which is moreover quasi-isomorphic to the dg-ring $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),B(\mathcal{A}))$ - where the dg-ring structure on the latter complex is composition of maps; see (2.2).

Proposition 2.3. *$\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ is a dg-ring.*

Proof. The product is clearly associative, and moreover it defines the structure of a unitary ring on $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$; the unit element of this ring is the augmentation map ϵ.

The proof of the Leibniz rule will be postponed to lemma 2.6 where we will use that the ring structure can also be obtained from the fact there is a diagonal map $\Delta : B(\mathcal{A})\to B(\mathcal{A})\otimes_{\mathcal{A}} B(\mathcal{A})$ of $\mathcal{A}^e$-modules which induces the product structure via the pullback. □

Let us start with this simple lemma

Lemma 2.4. *Assume that $\mathcal{P}\xrightarrow{\mu}\mathcal{A}$ is a cofibrant replacement of $\mathcal{A}$ in $\mathcal{A}^e-$*mod*. Then $\mathcal{P}\otimes_{\mathcal{A}}\mathcal{P}\xrightarrow{\mu\otimes\mu}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A}=\mathcal{A}$ is also a cofibrant replacement of $\mathcal{A}$ in $\mathcal{A}^e-$*mod.

Proof. In the category of left $\mathcal{A}$-modules $\mathcal{P}\xrightarrow{\mu}\mathcal{A}$ is a quasi-isomorphism of cofibrants, hence a homotopy equivalence; from this it follows that $\mathcal{P}\otimes_{\mathcal{A}}\mathcal{P}\xrightarrow{\mathrm{Id}_{\mathcal{P}}\otimes\mu}\mathcal{P}\otimes_{\mathcal{A}}\mathcal{A}$ is a quasi isomorphism. Similarly, $\mu\otimes\mathrm{Id}_{\mathcal{A}}$ is a quasi-isomorphism. Thus, $\mu\otimes\mu=\mu\otimes\mathrm{Id}_{\mathcal{A}}\circ\mathrm{Id}_{\mathcal{P}}\otimes\mu$ is a quasi-isomorphism.

To see that $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{P}$ is cofibrant take an exhaustive filtration $M_0 \subset M_1 \subset \ldots \subset \mathcal{P}$ with subquotients a direct sum of shifted copies of $\mathcal{A}^e$. Then the product filtration $M_0 \subset (M_0 \otimes_{\mathcal{A}} M_1) \oplus (M_1 \otimes_{\mathcal{A}} M_0) \subset \ldots \subset \mathcal{P}$ shows that $\mathcal{P}$ is cofibrant. □

The dg-ring structure on $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$ can be rephrased in terms of the diagonal map $\Delta : B(\mathcal{A}) \to B(\mathcal{A}) \otimes_{\mathcal{A}} B(\mathcal{A})$ defined by

$$\Delta(a_0|\ldots|a_{k+1}) = \sum_{j=0}^{k} (a_0|\ldots|a_j|1) \otimes (1|a_{j+1}|\ldots|a_{k+1}) \tag{2.6}$$

in the following way: for $\phi, \psi \in \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$ we see that

$$\phi \cdot \psi = \phi \otimes \psi \circ \Delta = \Delta^*(\phi \otimes \psi),$$

where $\phi \otimes \psi : B(\mathcal{A}) \otimes_{\mathcal{A}} B(\mathcal{A}) \to \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} = \mathcal{A}$ is given by $a \otimes a' \mapsto (-1)^{|a||\psi|}\phi(a)\psi(a')$. A priori, we know that this defines an associative product, but note that this associativity is also equivalent to the coassociativity of Δ.

The map Δ is coassociative in the sense that

$$\Delta \circ (\Delta \otimes \mathrm{Id}) = \Delta \circ (\mathrm{Id} \otimes \Delta) : B(\mathcal{A}) \to B(\mathcal{A}) \otimes_{\mathcal{A}} B(\mathcal{A}) \otimes_{\mathcal{A}} B(\mathcal{A}).$$

Abusing notations slightly, we write d, δ and D for the differentials on $B(\mathcal{A}) \otimes B(\mathcal{A})$ induced by Leibniz rule by the same differentials on $B(\mathcal{A})$.

Lemma 2.5. *The map $\Delta : B(\mathcal{A}) \to B(\mathcal{A}) \otimes_{\mathcal{A}} B(\mathcal{A})$ is a map of $\mathcal{A}^e$-modules.*

Proof. It is clear that Δ is A^e-linear. It remains to check that Δ commutes with $D = d + \delta$. This follows if we can prove that $d\Delta a = \Delta da$ and $\delta\Delta a = \Delta\delta a$ for $a = (a_0|\ldots|a_{k+1}) \in B(\mathcal{A})$. The straight forward computation is left to the reader. □

Now we are ready to prove that $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$ is a dg-ring,

Lemma 2.6. *The product in $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$ satisfies Leibniz rule.*

Proof. Let $\phi, \psi \in \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$. We must show that

$$d(\phi \cdot \psi) = (d\phi) \cdot \psi + (-1)^{|\phi|}\phi \cdot (d\psi) \tag{2.7}$$

We shall use Sweedlers notation $\Delta a = a_{(1)} \otimes a_{(2)}$, for $a \in B(\mathcal{A})$. We have

$$(d(\phi \cdot \psi))a = d(\phi \cdot \psi(a)) - (-1)^{|\phi|+|\psi|}\phi \cdot \psi(Da) =$$

$$d(\phi \otimes \psi(\Delta a)) - (-1)^{|\phi|+|\psi|}\phi \otimes \psi(\Delta Da) = [\text{ using } D\Delta = \Delta D] =$$

$$d(\phi \otimes \psi(a_{(1)} \otimes a_{(2)})) - (-1)^{|\phi|+|\psi|}\phi \otimes \psi(D(a_{(1)} \otimes a_{(2)})) \tag{2.8}$$

Using that $\phi \otimes \psi(a_{(1)} \otimes a_{(2)}) = (-1)^{|a_{(1)}||\psi|}\phi(a_{(1)})\psi a_{(2)}$ we see that (2.8) expands to

$$\begin{aligned}
&(-1)^{|a_{(1)}||\psi|}d(\phi(a_{(1)}))\psi(a_{(2)}) + (-1)^{|a_{(1)}||\psi|+|\phi|+|a_{(1)}|}\phi(a_{(1)})d(\psi(a_{(2)}))- \\
&\qquad (-1)^{|\phi|+|\psi|+|\psi|(|a_{(1)}|+1)}\phi(Da_{(1)})\psi(a_{(2)})- \\
&\qquad (-1)^{|\phi|+|\psi|+|\psi||a_{(1)}|+|a_{(1)}|}\phi(a_{(1)})\psi(Da_{(2)}) = \\
&\qquad (-1)^{|a_{(1)}||\psi|}\left\{d(\phi(a_{(1)})) - (-1)^{|\phi|}\phi(Da_{(1)})\right\}\psi(a_{(2)})+ \\
&\qquad (-1)^{|\phi|}(-1)^{|a_{(1)}|(|\psi|+1)}\phi(a_{(1)})\left\{d(\psi(a_{(2)})) - (-1)^{|\psi|}\psi(Da_{(2)})\right\} = \\
&(-1)^{|a_{(1)}||\psi|}(d\phi)(a_{(1)})\psi(a_{(2)}) + (-1)^{|\phi|}(-1)^{|a_{(1)}|(|\psi|+1)}\phi(a_{(1)})(d\psi)(a_{(2)}) = \\
&\qquad d\phi\otimes\psi(a_{(1)}\otimes a_{(2)}) + (-1)^{|\phi|}\phi\otimes d\psi(a_{(1)}\otimes a_{(2)}) = \\
&\qquad (d\phi\cdot\psi)a + (-1)^{|\phi|}(\phi\cdot d\psi)a
\end{aligned}$$

This proves (2.7). $\square$

Notice that the compositions

$$B(\mathcal{A}) \xrightarrow{\Delta} B(\mathcal{A}) \otimes_{\mathcal{A}} B(\mathcal{A}) \xrightarrow{1\otimes\epsilon} B(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A} = B(\mathcal{A})$$

$$B(\mathcal{A}) \xrightarrow{\Delta} B(\mathcal{A}) \otimes_{\mathcal{A}} B(\mathcal{A}) \xrightarrow{\epsilon\otimes 1} \mathcal{A} \otimes_{\mathcal{A}} B(\mathcal{A}) = B(\mathcal{A})$$

are the identity and that, by lemma 2.4 $B(\mathcal{A}) \otimes_{\mathcal{A}} B(\mathcal{A}) \xrightarrow{\epsilon\otimes\epsilon} \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} = \mathcal{A}$ is a cofibrant replacement for $\mathcal{A}$ (in the category of left $\mathcal{A}^e$-modules). Since the maps $1 \otimes \epsilon$ and $(1 \otimes \epsilon) \circ \Delta$ are quasi-isomorphisms, also Δ is a quasi-isomorphism.

Now note that since $\epsilon : B(\mathcal{A}) \to \mathcal{A}$ is a cofibrant replacement it induces a quasi-isomorphism of complexes

$$\epsilon_* : \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A})) \longrightarrow \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}), \tag{2.9}$$

but note that its kernel is not a right ideal and, consequently, ϵ_* cannot be a ring homomorphism.

However, we shall prove that

Proposition 2.7. *The map on cohomology induced by ϵ_* is multiplicative, hence gives a canonical ring isomorphism between*

$$H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}) \quad \textit{and} \quad H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A})).$$

Proof. We know that ϵ_* is a quasi-isomorphism so it suffices to show that $H^*(\epsilon_*)$ is multiplicative. For this purpose we shall start by construct two explicit quasi-inverses $\tilde{}$ and $\hat{}$ for ϵ_*.

Let us begin with $\tilde{}$. For $\phi \in \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$ we define

$$\phi^k = \phi|_{\mathcal{A}^{\otimes k+2}[k]} \in \mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{\otimes k+2}[k], \mathcal{A})$$

so that $\phi = \prod_{k=0}^{\infty} \phi^k$. For $i \geq 0$ define

$$\phi_i^k : \mathcal{A}^{k+i+2}[k+i] \to \mathcal{A}^{i+2}$$

$$(a_0|\dots|a_{k+i+1}) \mapsto \Big(\phi^k(a_0|\dots|a_k|1)|a_{k+1}|\dots|a_{k+i+1})\Big) \tag{2.10}$$

which assembled together define

$$\begin{array}{rcl} \widetilde{\phi}^j : \mathcal{A}^{j+2}[j] & \to & \mathcal{A}^2 \oplus \mathcal{A}^3[1] \oplus \cdots \oplus \mathcal{A}^{j+2}[j] \subset B(\mathcal{A}) \\ \mathbf{a} = (a_0|\dots|a_{j+1}) & \mapsto & \phi_0^k(\mathbf{a}) + \phi_1^{k-1}(\mathbf{a}) + \cdots + \phi_k^0(\mathbf{a}) \end{array}$$

and which assembled one step further gives

$$\widetilde{\phi} := \prod_{j=0}^{\infty} \widetilde{\phi}^j \in \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A})).$$

Note that $\epsilon_* \circ \widetilde{\ }$ is the identity map on $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$.

Lemma 2.8. *The map*

$$\widetilde{\ }: \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}) \to \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A}))$$

is a quasi isomorphism of complexes.

Proof. Let us show first that the map $\widetilde{\ }$commutes with the differentials. We denote the differentials on $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$ and on $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A}))$ by d (which is also the notation for the differential on $\mathcal{A}$ and for a component of the differential $D = d + \delta$ on $B(\mathcal{A})$). Thus, for $\phi \in \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$, and $a \in B(\mathcal{A})$ we have

$$(d\phi)(a) \stackrel{def}{=} d(\phi(a)) + \delta(\phi(a)) - (-1)^{|\phi|}\phi(da) \tag{2.11}$$

$$(d\widetilde{\phi})(a) \stackrel{def}{=} d(\widetilde{\phi}(a)) + \delta(\widetilde{\phi}(a)) - (-1)^{|\widetilde{\phi}|}(\widetilde{\phi}(da) + \widetilde{\phi}(\delta a)) \tag{2.12}$$

We must prove that

$$(d\widetilde{\phi})(a) = \widetilde{d\phi}(a) \tag{2.13}$$

We can assume that

$$\phi = \phi^k : \mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A} \to \mathcal{A},$$

for some k and that $a = (a_0|\dots|a_{k+i+1})$ for some i. We have

$$\begin{aligned}
d(\widetilde{\phi}(a)) - (-1)^{|\phi|}\widetilde{\phi}(da) = &(d(\phi(a_0|\dots|a_k|1))|a_{k+1}|\dots|a_{k+i+1}) \\
&- \sum_{j=1}^{i}(-1)^{|\phi|+\varepsilon_{k+j}}(\phi(a_0|\dots|a_k|1)|a_{k+1}|\dots|da_{k+j}|\dots|a_{k+i+1})) \\
&+ (-1)^{|\phi|+\varepsilon_{k+i}}(\phi(a_0|\dots|a_k|1)|a_{k+1}|\dots|da_{k+i+1})) \\
&- (-1)^{|\phi|}(\phi(d(a_0|\dots|a_k|1))|a_{k+a}|\dots|a_{k+i+1}) \\
&+ \sum_{j=1}^{i}(-1)^{|\phi|+\varepsilon_{k+j}}(\phi(a_0|\dots|a_k|1)|a_{k+1}|\dots|da_{k+j}|\dots|a_{k+i+1})) \\
&- (-1)^{|\phi|+\varepsilon_{k+i}}(\phi(a_0|\dots|a_k|1)|a_{k+1}|\dots|da_{k+i+1})) \\
=&(d(\phi(a_0|\dots|a_k|1))|a_{k+1}|\dots|a_{k+i+1}) \\
&- (-1)^{|\phi|}(\phi(d(a_0|\dots|a_k|1))|a_{k+1}|\dots|a_{k+i+1}) =: A
\end{aligned}$$

where the last equality holds since line 2 and 5 and line 3 and 6 cancel. Similarly, we have

$$\begin{aligned}
&\delta(\widetilde{\phi}(a)) - (-1)^{||\phi||}\widetilde{\phi}(\delta a) \\
&\quad=\delta(\phi(a_0|\dots|a_k|1)|a_{k+1}|\dots|a_{k+i+1}) \\
&\quad\quad - (-1)^{|\phi|}\widetilde{\phi}(\delta(a_0|\dots|a_{k+i+1}) \\
&\quad=(-1)^{|\phi|+\varepsilon_k}(\phi(a_0|\dots|a_k|1)a_{k+1}|a_{k+2}|\dots|a_{k+i+1}) \\
&\quad\quad + \sum_{j=1}^{i-1}(-1)^{|\phi|+\varepsilon_{k+j}}(\phi(a_0|\dots|a_k|1)|a_{k+1}|\dots|a_{k+j}a_{k+j+1}|\dots|a_{k+i+1}) \\
&\quad\quad - (-1)^{|\phi|+\varepsilon_{k+i}}(\phi(a_0|\dots|a_k|1)|a_{k+1}|\dots|a_{k+i}a_{k+i+1}) \\
&\quad\quad - \sum_{j=0}^{k}(-1)^{|\phi|+\varepsilon_j}(\phi(a_0|\dots|a_ja_{j+1}|\dots|a_{k+1}|1)|a_{k+2}|\dots|a_{k+i+1}) \\
&\quad\quad - \sum_{j=1}^{i-1}(-1)^{||\phi||+\varepsilon_{k+j}}(\phi(a_0|\dots|a_k|1)|a_{k+1}|\dots|a_{k+j}a_{k+j+1}|\dots|a_{k+i+1}) \\
&\quad\quad + (-1)^{|\phi|+\varepsilon_{k+i}}(\phi(a_0|\dots|a_k|1)|a_{k+1}|\dots|a_{k+i}a_{k+i+1}) \\
&\quad= - \sum_{j=0}^{k}(-1)^{|\phi|+\varepsilon_j}(\phi(a_0|\dots|a_ja_{j+1}|\dots|a_{k+1}|1)|a_{k+2}|\dots|a_{k+i+1}) \\
&\quad\quad (-1)^{|\phi|+\varepsilon_k}(\phi(a_0|\dots|a_k|a_{k+1})|a_{k+2}|\dots|a_{k+i+1}) \\
&\quad= - (-1)^{|\phi|}(\phi(\delta(a_0|\dots|a_{k+1}|1))|a_{k+2}|\dots|a_{k+i+1})) =: B
\end{aligned}$$

It follows from (2.11) and the definition of $\widetilde{\ }$ that $A+B = \widetilde{d\phi}(a)$. Then (2.12) proves (2.13). As the composition $\epsilon_* \circ \widetilde{\ } = \mathrm{Id}$ and ϵ_* are quasi-isomorphisms it follows that $\widetilde{\ }$ is a quasi-isomorphism. $\square$

Next, let us define

$$\widehat{\ }: \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}) \to \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A}))$$

as follows: Let $\phi \in \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$ and let ϕ_i^k be defined as in (2.10). Define

$$\begin{aligned} \widehat{\phi}^j : \mathcal{A}^{j+2}[j] &\to \mathcal{A}^2 \oplus \mathcal{A}^3[1] \oplus \cdots \oplus \mathcal{A}^{j+2}[j] \subset B(\mathcal{A}) \\ \mathbf{a} = (a_0|\dots|a_{j+1}) &\mapsto \sum_{i=0}^{k} (-1)^{|\phi|+\varepsilon_i} \phi_i^{k-i}(\mathbf{a}) \end{aligned}$$

and finally define

$$\widehat{\phi} := \prod_{j=0}^{\infty} \widehat{\phi}^j \in \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A})).$$

Just as in the proof of lemma 2.8 one can show that $\widehat{\ }$ is a morphisms of complexes, we omit to repeat the proof. The composition $\epsilon_* \circ \widehat{\ }$ is the identity on $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$, because for $\psi \in \mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{l+2}[l], \mathcal{A})$ we have

$$\begin{aligned} (\epsilon_* \circ \widehat{\phi})(a_0|\dots|a_{l+1}) &= (-1)^{|a_0||\phi|} \epsilon(a_0|\phi(1|a_1|\dots|a_{l+1})) \\ &= (-1)^{|a_0||\phi|} a_0 \psi(1|a_1|\dots|a_{l+1}) \\ &= \phi(a_0|a_1|\dots|a_{l+1}) \end{aligned}$$

Since ϵ_* is a quasi-isomorphism it follows that $\widehat{\ }$ is a quasi-isomorphism too.

Now we are ready to finish the proof of proposition 2.7. Take the two maps

$$\phi \in \mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{k+2}[k], \mathcal{A}) \quad \text{and} \quad \psi \in \mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{l+2}[l], \mathcal{A})$$

and consider $\widetilde{\phi}$ and $\widehat{\psi}$ in $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A})B(\mathcal{A}))$. We claim that

$$\epsilon_* \circ \widetilde{\phi} \circ \widehat{\psi} = \phi \cdot \psi.$$

This holds because

$$\begin{aligned} &\epsilon_* \circ \widetilde{\phi} \circ \widehat{\psi}(a_0|\dots|a_{k+l+1}) \\ &\qquad = \epsilon\left(\widetilde{\phi}\left((-1)^{|\psi|\varepsilon_k}(a_0|\dots|a_k|\psi(1|a_{k+1}|\dots|a_{k+l+1}))\right)\right) \\ &\qquad = (-1)^{|\psi|\varepsilon_k} \epsilon\left(\phi(a_0|\dots|a_k|1)|\psi(1|a_{k+1}|\dots|a_{k+l+1})\right) \\ &\qquad = (-1)^{|\psi|\varepsilon_k} \phi(a_0|\dots|a_k|1)\psi(1|a_{k+1}|\dots|a_{k+l+1}) \\ &\qquad = (\phi \cdot \psi)(a_0|\dots|a_{k+l+1}). \end{aligned}$$

This implies that $H^*(\epsilon_*)$ is multiplicative, because given any cohomology classes $[f], [g] \in H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), B(\mathcal{A}))$ we have, since $\widetilde{\ }$ and $\widehat{\ }$ are quasi-isomorphisms, that there exists $\phi, \psi \in \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A})$ such that $[f] = [\widetilde{\phi}]$

and $[g]=[\widehat{\psi}]$. Thus $H^*(\epsilon_*)([f]\circ[g])=$

$$H^*(\epsilon_*)([\widetilde{\phi}]\circ[\widehat{\psi}])=H^*(\epsilon_*)([\widetilde{\phi}\circ\widehat{\psi}])=[\epsilon_*(\widetilde{\phi}\circ\widehat{\psi})]=[\phi\cdot\psi]=$$
$$[\epsilon_*(f)\cdot\epsilon_*(g)]=H^*(\epsilon_*)([f])\cdot H^*(\epsilon_*)([g])$$

This shows that $H^*(\epsilon_*)$ is multiplicative. □

To finish this section let us show that the construction performed with the diagonal map for $B(\mathcal{A})$ can be generalized to other cofibrant replacements of $\mathcal{A}$:

Proposition 2.9. *Let $\mathcal{P}\xrightarrow{\mu}\mathcal{A}$ be a cofibrant replacement of $\mathcal{A}$ and furthermore assume that there exist a homomorphism of $\mathcal{A}^e$-modules*

$$\Delta_{\mathcal{P}}:\mathcal{P}\to\mathcal{P}\times_{\mathcal{A}}\mathcal{P}$$

such that the compositions

$$\mathcal{P}\xrightarrow{\Delta_{\mathcal{P}}}\mathcal{P}\otimes_{\mathcal{A}}\mathcal{P}\xrightarrow{1\otimes\mu}\mathcal{A}\otimes_{\mathcal{A}}\mathcal{P}=\mathcal{P} \tag{2.14}$$
$$\mathcal{P}\xrightarrow{\Delta_{\mathcal{P}}}\mathcal{P}\otimes_{\mathcal{A}}\mathcal{P}\xrightarrow{\mu\otimes 1}\mathcal{P}\otimes_{\mathcal{A}}\mathcal{A}=\mathcal{P}$$

are both the identity, and that the maps $\Delta_{\mathcal{P}}$ is coassociative. Then the product structure defined by the map

$$\begin{array}{rcl}\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})\times\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A}) & \to & \mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})\\ \phi\times\psi & \mapsto & \Delta_{\mathcal{P}}^*(\phi\otimes\psi)\end{array}$$

makes $\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})$ into an associative dg-ring which induces an associative ring structure on $H^*\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})$. This ring is canonically isomorphic to $HH^*(\mathcal{A},\mathcal{A})$.

Proof. The same argument as in lemma 2.6 shows that $\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})$ is a dg-ring; its associativity follows from the coassociativity of $\Delta_{\mathcal{P}}$. The fact that μ is a unit follows from the compositions of (2.14).

Now, since $B(\mathcal{A})$ and $\mathcal{P}$ are cofibrant, there exists a quasi-isomorphism of $\mathcal{A}^e$-modules $\alpha:B(\mathcal{A})\to\mathcal{P}$, unique up to homotopy such that $\mu\circ\alpha=\epsilon$. This gives a map $\phi\mapsto\alpha^*(\phi)$ from $\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})$ to $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$. Taking cohomology we get a graded group isomorphism

$$H^*\mathcal{H}om_{\mathcal{A}^e}(\mathcal{P},\mathcal{A})\to H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A}),\ \phi\mapsto\alpha^*(\phi). \tag{2.15}$$

which is independent of the choice of α.

Now consider the maps

$$\Delta_P\circ\alpha,\alpha\otimes\alpha\circ\Delta:B(\mathcal{A})\to\mathcal{P}\otimes_{\mathcal{A}}\mathcal{P}.$$

We have that $\mu\otimes\mu:\mathcal{P}\otimes_{\mathcal{A}}\mathcal{P}\to\mathcal{A}$ is a cofibrant replacement and that

$$\mu\otimes\mu\circ\Delta_P\circ\alpha=\mu\otimes\mu\circ\alpha\otimes\alpha\circ\Delta=\epsilon\otimes\epsilon.$$

We conclude that $\Delta_P\circ\alpha$ and $\alpha\circ\Delta$ are homotopic. Hence they induce the same map on cohomology which implies that (2.15) is a ring homomorphism. As $H^*\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ is canonically isomorphic to $HH^*(\mathcal{A},\mathcal{A})$, the proposition follows.

□

3. Hochschild cohomology for the smash product of a group and a dg-ring

In this section we will describe how to calculate the Hochschild cohomology for the smash product of a group and a dg-ring. The main construction of this section consists of an explicit cofibrant replacement from which the results that we claim follow clearly, and which gives an alternative explanation of the results of [23] over the Hochschild cohomology of crossed products over commutative rings.

In this section G is a finitely generated discrete group.

Let $\mathcal{A} = (A, d_{\mathcal{A}})$ be a dg-ring together with a group homomorphism $\sigma : G \to \mathrm{Aut}_{dg-Rings}(A)$, we write $ga := g(a) := \sigma(g)(a)$, for $a \in A$, $g \in G$. We refer to such an $\mathcal{A}$ as a G-module dg-ring (this name is a dg-analog of the term module-algebra.)

Definition 3.1. Let $\mathcal{A}$ be a G-module dg-ring. The smash product gives a dg-ring that is a principal object of study in this paper

$$\mathcal{A}\#G := \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}G$$

with multiplication given on generators by

$$(x \otimes g)(y \otimes h) := xg(y) \otimes gh$$

and the differential is $d_{\mathcal{A}} \otimes 1$.

Note that $\mathcal{A}$ is a sub-dg-ring of $\mathcal{A}\#G$ by the map $x \mapsto x \otimes 1_G$, and $\mathbb{Z}G$ is also a sub-dg-ring of $\mathcal{A}\#G$ via the map $g \mapsto 1_{\mathcal{A}} \otimes g$.

In this section we will construct an explicit cofibrant replacement for $\mathcal{A}\#G$ as a $(\mathcal{A}\#G)^e$-module, that together with an explicit diagonal map, will be the main tools to show that if we consider $\mathcal{A}\#G$ as a G-module-dg-ring by the action $g \cdot (x \otimes h) \mapsto g(x) \otimes ghg^{-1}$, then we have that

Theorem 3.2. *There are isomorphisms of graded groups*

$$HH_*((\mathcal{A}\#G, \mathcal{A}\#G) \cong \mathrm{Tor}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}\#G)$$

$$HH^*(\mathcal{A}\#G, \mathcal{A}\#G) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{RH}om_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G))$$

defined in an appropriate way such that the second isomorphism becomes one of graded rings.

The explicit ring structure on the right hand side of the second isomorphism will be explained in the proof of the theorem, and it is further emphasized in sections 3.3.1 and 3.3.2. This ring structure will be of use when we will study the string topology for orbifolds and its relation with Hochschild cohomology of the dg-rings of singular cochains.

3.1. **Cofibrant replacement for $\mathcal{A}\#G$.** Let us consider the cofibrant replacements constructed via the bar construction (see section 2.2.1) for the dg-ring $\mathcal{A}$ as an $\mathcal{A}^e$-module and for the ring $\mathbb{Z}G$ as a $\mathbb{Z}G^e$-module

$$B(\mathcal{A}) \xrightarrow{\epsilon} \mathcal{A} \qquad B(\mathbb{Z}G) \to \mathbb{Z}G.$$

If we consider the isomorphism

$$\begin{array}{rcl} (\mathbb{Z}G)^{k+2} & \to & (\mathbb{Z}G)^{k+2} \\ (g_0|g_1|\dots|g_{k+1}) & \mapsto & (g_0|g_0g_1|g_0g_1g_2|\dots|g_0\dots g_kg_{k+1}) \end{array}$$

then by transportation of structures we can change $B(\mathbb{Z}G)$ by an alternative cofibrant replacement $\overline{B}(\mathbb{Z}G)$ of $\mathbb{Z}G$ defined as follows: as a $\mathbb{Z}$-graded module we have that

$$\overline{B}(\mathbb{Z}G) = \bigoplus_{k=0}^{\infty} (\mathbb{Z}G)^{k+2}[k]$$

the differential becomes

$$\delta(h_0|\dots|h_{k+1}) = \sum_{j=0}^{k} (-1)^j (h_0|\dots|\widehat{h_j}|\dots|h_{k+1}),$$

the $\mathbb{Z}G^e$-module structure is

$$(g\otimes k)(h_0|\dots|h_{k+1}) = (gh_0|gh_1|gh_2|\dots|gh_{k+1}k)$$

and the $\mathbb{Z}G^e$-module homomorphism $\bar{\epsilon} : \overline{B}(\mathbb{Z}G) \to \mathbb{Z}G$ is

$$\bar{\epsilon}(h_0|h_1) = h_1 \quad \text{and} \quad \bar{\epsilon}(h_0|\dots|h_{k+1}) = 0 \text{ for } k > 1.$$

Notice that for $\overline{B}(\mathbb{Z}G)$ the diagonal map defined in (2.6) becomes

$$\begin{array}{rcl} \overline{B}(\mathbb{Z}G) & \to & \overline{B}(\mathbb{Z}G) \otimes_{\mathbb{Z}G} \overline{B}(\mathbb{Z}G) \\ (h_0|\dots|h_{k+1}) & \mapsto & \displaystyle\sum_{j=0}^{k} (h_0|\dots|h_j|1) \otimes_{\mathbb{Z}G} (h_j|h_{j+1}|\dots|h_{k+1}). \end{array} \tag{3.1}$$

Lemma 3.3. *The tensor product $B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G)$ can be endowed with the structure of a $\mathcal{A}\#G^e$-module structure thus making*

$$\epsilon\otimes\bar{\epsilon} : B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G) \to \mathcal{A}\#G$$

a cofibrant replacement for $\mathcal{A}\#G$ as an $\mathcal{A}\#G^e$-module.

Proof. Let us denote the elements in $\mathcal{A}\#G^e$ by $(a\otimes g|b\otimes k)$ where $a\otimes g$ and $b\otimes k$ belong to $\mathcal{A}\#G$. The $\mathcal{A}\#G^e$-module structure of $B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G)$ is defined as

$$(a\otimes g|b\otimes k)\left((x_0|\dots|x_{k+1})\otimes(h_0|\dots|h_{l+1})\right) =$$

$$(ag(x_0)|g(x_1)|\dots|g(x_k)|g(x_{k+1})gh_{l+1}(b)) \otimes (gh_0|gh_1\dots|gh_l|gh_{l+1}k).$$

It is a simple calculation to show that indeed the previous structure is a $\mathcal{A}\#G$-module structure on $B(\mathcal{A})\otimes\overline{B}(\mathbb{Z}G)$.

To show that $\epsilon \otimes \bar{\epsilon} : B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G) \to \mathcal{A}\#G$ is a morphism of $\mathcal{A}\#G^e$-modules we need only to concentrate our attention to the elements in $\mathcal{A}^2 \otimes_{\mathbb{Z}} \mathbb{Z}G^2$ and that follows from the commutativity of the following diagram

$$\begin{tikzcd}
(x_0|x_1) \otimes (h_0, h_1) \arrow[r, "(a\otimes g|b\otimes k)"] \arrow[d, "\epsilon\otimes\bar{\epsilon}"] & (ag(x_0)|g(x_1)gh_1(b)) \otimes (gh_0|gh_1k) \arrow[d, "\epsilon\otimes\bar{\epsilon}"] \\
(x_0x_1|h_1) \arrow[r, "(a\otimes g|b\otimes k)"] & (ag(x_0x_1)gh_1(b)|gh_1k)
\end{tikzcd}$$

The fact that the map $\epsilon \otimes \bar{\epsilon}$ is a quasi-isomorphism follows from the facts that ϵ and $\bar{\epsilon}$ are quasi-isomorphisms. Now we are left to prove the cofibrant condition. For it consider filtration defined for $q \geq 0$

$$F^{2q} := \bigoplus_{p=0}^{q} \bigoplus_{j=0}^{p} \left((\mathcal{A} \otimes \mathcal{A}^{\otimes j} \otimes \mathcal{A}) \otimes (\mathbb{Z}G)^{p-j+2} \right) [p]$$

and

$$F^{2q+1} := F^{2q} \oplus \bigoplus_{p=0}^{q} \left((\mathcal{A} \otimes W^{p+1} \otimes \mathcal{A}) \otimes (\mathbb{Z}G)^{q-p+2} \right) [q+1]$$

where W^p is the $\mathcal{A}^e$-submodule of $\mathcal{A}^p$ defined as the kernel of the differential $d : \mathcal{A}^p \to \mathcal{A}^p$ (as in the proof of lemma 2.1).

The subquotients of the filtration are isomorphic to the $\mathcal{A}\#G$-modules

$$F^{2q+1}/F^{2q} \cong \bigoplus_{p=0}^{q} \left((\mathcal{A} \otimes W^{p+1} \otimes \mathcal{A}) \otimes (\mathbb{Z}G)^{q-p+2} \right) [q+1]$$

$$F^{2q}/F^{2q-1} \cong \bigoplus_{j=0}^{q} \left((\mathcal{A} \otimes \mathcal{A}^{\otimes j}/W^j \otimes \mathcal{A}) \otimes (\mathbb{Z}G)^{q-j+2} \right) [q]$$

where in both cases the induced differential is only different from zero on the components of $\mathcal{A}$ on the far left and on the far right. As the $\mathbb{Z}G^e$-modules $(\mathbb{Z}G)^{q-p+2}$ are all free, it follows that the subquotients F^{2q+1}/F^{2q}, F^{2q}/F^{2q-1} are summands of a direct sum of shifted copies of $\mathcal{A}\#G^e$, therefore $B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G)$ is cofibrant.

□

Now let us define a diagonal map for $B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G)$:

$$B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G) \to B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G) \otimes_{\mathcal{A}\#G} B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G) \tag{3.2}$$

$$(x_0| \dots |x_{k+1}) \otimes (h_0| \dots |h_{l+1}) \mapsto$$

$$\sum_{i=0}^{k} \sum_{j=0}^{l} (x_0|...|x_i|1) \otimes (h_0|...|h_j|1) \otimes_{\mathcal{A}\#G} (1|x_{i+1}|...|x_{k+1}) \otimes (h_j|h_{j+1}|...|h_{l+1})$$

that is just the juxtaposition of the diagonals maps for $B(\mathcal{A})$ and $\overline{B}(\mathbb{Z}G)$ defined in (2.6) and (3.1) respectively. It follows then that the diagonal map

for $B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G)$ satisfies the hypothesis for Proposition 2.9 and therefore it will induce the ring structure on the Hochschild cohomology of $\mathcal{A}\#G$.

We are now ready to prove the main theorem of this section.

3.2. **Proof of Theorem 3.2.** We have that $B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G) \to \mathcal{A}\#G$ is a cofibrant replacement for $\mathcal{A}\#G$ as an $\mathcal{A}\#G^e$-module. Therefore we have the isomorphisms

$$HH_*(\mathcal{A}\#G, \mathcal{A}\#G) = H^*\left(\left(B(\mathcal{A}) \otimes \overline{B}(\mathbb{Z}G)\right) \otimes_{\mathcal{A}\#G^e} \mathcal{A}\#G\right)$$

and

$$HH^*(\mathcal{A}\#G, \mathcal{A}\#G) = H^*\mathcal{H}om_{\mathcal{A}\#G^e}\left(B(\mathcal{A}) \otimes \overline{B}(\mathbb{Z}G), \mathcal{A}\#G\right).$$

Let us consider the sub-dg-rings $\mathbb{Z}G^e \subset \mathcal{A}\#G^e$ with $g \otimes k \mapsto (1 \otimes g | 1 \otimes k)$ and $\mathcal{A}^e \subset \mathcal{A}\#G^e$ with $x_0 \otimes x_1 \mapsto (x_0 \otimes 1 | x_1 \otimes 1)$, and note that as such $\mathcal{A}^e$ and $\mathbb{Z}G^e$ generate the dg-ring $\mathcal{A}\#G^e$. The sub-dg-ring $\mathcal{A}^e$ acts trivially on the component $\overline{B}(\mathbb{Z}G)$ of $B(\mathcal{A}) \otimes \overline{B}(\mathbb{Z}G)$ therefore we have an isomorphism

$$\left(B(\mathcal{A}) \otimes \overline{B}(\mathbb{Z}G)\right) \otimes_{\mathcal{A}\#G^e} \mathcal{A}\#G \cong \overline{B}(\mathbb{Z}G) \otimes_{\mathbb{Z}G^e} \left(B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}\#G\right)$$

where the induced action of $g \otimes k \in \mathbb{Z}G^e$ into $B(\mathcal{A})$ is diagonal on g and trivial on k, i.e.

$$(g \otimes k)(x_0 | \dots | x_{k+1}) \mapsto (g(x_0) | \dots | g(x_{k+1})).$$

The previous argument applies also for the $\mathcal{H}om$ functor and therefore we have

$$\mathcal{H}om_{\mathcal{A}\#G^e}\left(B(\mathcal{A}) \otimes \overline{B}(\mathbb{Z}G), \mathcal{A}\#G\right) \cong \mathcal{H}om_{\mathbb{Z}G^e}\left(\overline{B}(\mathbb{Z}G), \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)\right).$$

Let us now see more carefully the functors

$$\overline{B}(\mathbb{Z}G) \otimes_{\mathbb{Z}G^e} \_ \qquad \text{and} \qquad \mathcal{H}om_{\mathbb{Z}G^e}\left(\overline{B}(\mathbb{Z}G), \_\right).$$

Note that the action of the elements of the form $(1 \otimes k)$ is by multiplication on the right, and this action is free on $\overline{B}(\mathbb{Z}G)$. Note also that we could generate the ring $\mathbb{Z}G^e$ by the subrings $1 \otimes \mathbb{Z}G$ and the image of the diagonal homomorphism $\Delta : \mathbb{Z}G \to \mathbb{Z}G^e$, $\Delta(g) = g \otimes g^{-1}$. Therefore we could restrict our attention to the sub-complex $\overline{B}_G(\mathbb{Z})$ of $\overline{B}(\mathbb{Z}G)$

$$\overline{B}_G(\mathbb{Z}) = \{(h_0 | \dots | h_{k+1}) \in \overline{B}(\mathbb{Z}G) | h_{k+1} = 1\}$$

consisting of the elements which end in 1, disregarding the module structure of the subring $1 \otimes \mathbb{Z}G$ and considering it as a $\mathbb{Z}G$-module given by the action induced by its image under the diagonal map in $\mathbb{Z}G^e$.

Note that the $\mathbb{Z}G$ module structure $\overline{B}_G(\mathbb{Z})$ becomes a diagonal action

$$\begin{aligned} g \cdot (h_0 | \dots | h_k | 1) &:= (g \otimes g^{-1})(h_0 | \dots h_k | 1) \\ &= (gh_0 | \dots | gh_k | g1g^{-1}) \\ &= (gh_0 | \dots | gh_k | 1) \end{aligned}$$

and that $\overline{B}_G(\mathbb{Z})$ becomes a cofibrant replacement for $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module.

Thus we have the isomorphisms of complexes

$$\overline{B}(\mathbb{Z}G) \otimes_{\mathbb{Z}G^e} (B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}\#G) \cong \overline{B}_G(\mathbb{Z}) \otimes_{\mathbb{Z}G} (B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}\#G)$$

$$\mathcal{H}om_{\mathbb{Z}G^e}\left(\overline{B}(\mathbb{Z}G), \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)\right) \cong \mathcal{H}om_{\mathbb{Z}G}\left(\overline{B}_G(\mathbb{Z}), \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)\right)$$

where the $\mathbb{Z}G$-module structure of $B(\mathcal{A})$ is given by the diagonal action and the $\mathbb{Z}G$-module structure of $\mathcal{A}\#G$ is given by the natural action on $\mathcal{A}$ and by conjugation on G, i.e.

$$g \cdot (x \otimes k) = (1 \otimes g)(x \otimes k)(1 \otimes g^{-1}) = (g(x) \otimes gkg^{-1}).$$

We can therefore endow $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)$ with the structure of a $\mathbb{Z}G$-module as follows: for $f \in \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)$ and $g \in G$ we have

$$(gf)(x_0|\dots|x_{k+1}) := g^{-1}(f(g(x_0)|\dots|g(x_{k+1}))). \tag{3.3}$$

Hence we can conclude that there are isomorphisms

$$\begin{aligned} HH_*(\mathcal{A}\#G, \mathcal{A}\#G) &= H^*\left(\overline{B}_G(\mathbb{Z}) \otimes_{\mathbb{Z}G} (B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}\#G)\right) \\ &= \mathrm{Tor}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}\#G) \end{aligned}$$

$$\begin{aligned} HH^*(\mathcal{A}\#G, \mathcal{A}\#G) &= H^*\mathcal{H}om_{\mathbb{Z}G}\left(\overline{B}_G(\mathbb{Z}), \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)\right) \\ &= \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{RH}om_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)). \end{aligned}$$

3.2.1. We are now left to describe the dg-ring structure of

$$\mathcal{H}om_{\mathbb{Z}G}\left(\overline{B}_G(\mathbb{Z}), \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)\right)$$

that will induce the ring structure on the Hochschild cohomology. The dg-ring structure is induced from the diagonal map for $B(\mathcal{A}) \otimes_{\mathbb{Z}} \overline{B}(\mathbb{Z}G)$ that was defined in (3.2) and from this it follows that for

$$\phi, \psi \in \mathcal{H}om_{\mathbb{Z}G}\left(\overline{B}_G(\mathbb{Z}), \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)\right)$$

we have that

$$\begin{aligned} &\phi \cdot \psi(h_0|\dots|h_k|1)(x_0|\dots|x_{l+1}) = \\ &\sum_{i=0}^{k}\sum_{j=0}^{l} \left(\phi(h_0|\dots|h_i|1)(x_0|\dots|x_j|1)\right)\left(\psi(h_i|\dots|h_k|1)(1|x_{j+1}|\dots|x_{l+1})\right). \end{aligned} \tag{3.4}$$

This ends the proof of theorem 3.2.

Note that the $\mathbb{Z}G$-module structure on $\mathcal{A}\#G$ is given by $g \cdot (x \otimes k) = (g(a), gkg^{-1})$. Therefore we can split $\mathcal{A}\#G$ into $\mathbb{Z}G$-modules in the following way

$$\mathcal{A}\#G \cong \bigoplus_{T \in [G]} \mathcal{A}_T \qquad \text{with} \qquad \mathcal{A}_T = \bigoplus_{h \in T} \mathcal{A}_h \tag{3.5}$$

where $[G]$ is the set of conjugacy classes of elements in G and $\mathcal{A}_h$ is the subset of $\mathcal{A}\#G$ of elements of the form $x \otimes h$,

$$\mathcal{A}_h := \{x \otimes h \in \mathcal{A}\#G | x \in \mathcal{A}\}. \tag{3.6}$$

Let us choose one element of G for each conjugacy class in $[G]$, and let us denote this set of representatives by $\langle G \rangle$. Then we have

Corollary 3.4. *There are isomomorphisms of graded groups*

$$HH_*(\mathcal{A}\#G, \mathcal{A}\#G) \cong \bigoplus_{g \in \langle G \rangle} \mathrm{Tor}_{\mathbb{Z}C_g}(\mathbb{Z}, B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}_g)$$

$$HH^*(\mathcal{A}\#G, \mathcal{A}\#G) \cong \bigoplus_{g \in \langle G \rangle} \mathrm{Ext}_{\mathbb{Z}C_g}(\mathbb{Z}, \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_g))$$

where C_g is the centralizer of g in G and $\mathcal{A}_g$ is viewed as a $\mathbb{Z}C_g$ module.

Proof. Both isomorphisms are proved by the same argument; let us prove the second one. As a $\mathbb{Z}G$-module have the isomorphism

$$\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G) \cong \bigoplus_{T \in [G]} \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_T)$$

and therefore we have that

$$\begin{aligned} \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)) &\cong \bigoplus_{T \in [G]} \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_T)) \\ &\cong \bigoplus_{g \in \langle G \rangle} \mathrm{Ext}^*_{\mathbb{Z}C_g}(\mathbb{Z}, \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_g)). \end{aligned}$$

□

Corollary 3.5. *If $\mathcal{A}$ is a free $\mathbb{Z}G$-module-dg-ring, then we have isomorphisms of graded groups*

$$HH_*(\mathcal{A}\#G, \mathcal{A}\#G) \cong \bigoplus_{g \in \langle G \rangle} H^* \left((B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}_g)^{C_g} \right)$$

$$HH^*(\mathcal{A}\#G, \mathcal{A}\#G) \cong \bigoplus_{g \in \langle G \rangle} H^* \left((\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}_g))^{C_g} \right).$$

Proof. The isomorphisms follow form corollary 3.4 and the fact that for free modules the invariants and the coinvariants are isomorphic. □

3.3. **Further results.** We end up this section with some results that follow from the proof of theorem 3.2.

3.3.1. The diagonal map of $B(\mathcal{A})$ defined on (2.6) endows the complex

$$\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)$$

with the structure of a dg-ring which is moreover a $\mathbb{Z}G$-module dg-ring. Following the notation of (3.6) if we take two functions

$$\phi \in \mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{k+2}[k], \mathcal{A}_g), \quad \psi \in \mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{l+2}[l], \mathcal{A}_h)$$

then its product $\phi \cdot \psi$ lives in $\mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{k+l+2}[k+l], \mathcal{A}_{gh})$ and is defined as

$$\begin{aligned}(\phi \cdot \psi)(a_0|\dots|a_{k+l+2})\\ &=(-1)^{|\psi|\varepsilon_k}(\phi_g(a_0|\dots|a_k|1)\otimes g)(\psi_h(1|a_{k+1}|\dots|a_{k+l+1})\otimes h)\\ &=(-1)^{|\psi|\varepsilon_k}\phi_g(a_0|\dots|a_k|1)g(\psi_h(1|a_{k+1}|\dots|a_{k+l+1}))\otimes gh\end{aligned}$$

where we use the convention

$$\phi(\mathbf{a}) = \sum_g (\phi_g(\mathbf{a}) \otimes g).$$

The $\mathbb{Z}G$-module structure of $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)$ is defined in (3.3) and it is easy to check that for $k \in G$ we $(k\phi)(k\psi) = k(\phi \cdot \psi)$.

We therefore have that $\mathrm{Ext}^*_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)$ becomes a $\mathbb{Z}G$-module ring.

3.3.2. Whenever W is a $\mathbb{Z}G$-module ring, the complex $\mathcal{H}om_{\mathbb{Z}G}(\overline{B}_G(\mathbb{Z}), W)$ can be endowed with the structure of a dg-ring with a product structure induced by the formula in (3.4), i.e. for $\alpha, \beta \in \mathcal{H}om_{\mathbb{Z}G}(\overline{B}_G(\mathbb{Z}), W)$ we have that

$$\alpha \cdot \beta \ (h_0|\dots|h_l|1) = \sum_{j=0}^{l} \alpha(h_0|\dots|h_j|1)\beta(h_j|\dots h_l|1).$$

This dg-ring structure makes $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, W)$ into a ring. In the case that $W = \mathbb{Z}$, the ring $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ is isomorphic to the ring $H^*(BG, \mathbb{Z})$.

3.3.3. If we filter both complexes

$$\overline{B}_G(\mathbb{Z}) \otimes_{\mathbb{Z}G} (B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}\#G)$$
$$\mathcal{H}om_{\mathbb{Z}G}\left(\overline{B}_G(\mathbb{Z}), \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G)\right)$$

by the degree of the elements in $\overline{B}_G(\mathbb{Z})$ then we get spectral sequences which abut to the Hochschild homology and cohomology respectively and whose second page is given by

$$E^2 = \mathrm{Tor}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Tor}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)) \Rightarrow HH_*(\mathcal{A}\#G, \mathcal{A}\#G)$$

and

$$E^2 = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Ext}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G)) \Rightarrow HH^*(\mathcal{A}\#G, \mathcal{A}\#G).$$

In the case of the Hochschild cohomology ring, the spectral sequence is a sequence of rings where

$$\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Ext}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}\#G))$$

has the ring structure explained in 3.3.1 and 3.3.2.

4. Loops on quotient spaces

This section is devoted to show the relation between the complexes

$$B(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}\#G \qquad \text{and} \qquad \mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}), \mathcal{A}\#G),$$

and the cochains and chains repectively for the loops on the groupoid $[X/G]$ whenever we take $\mathcal{A} = C^*X$.

4.1. **Loops on $[X/G]$.** Let X be a connected CW-complex of finite type and G a discrete group acting on X. Denote by $[X/G]$ the action groupoid whose objects are X and whose morphisms are $X \times G$ with $s(x,g) = x$ and $t(x,g) = xg$.

The loops on $[X/G]$ can be understood as the groupoid whose objects are the functors from the groupoid $\mathbb{R}/\mathbb{Z}$ to $[X/G]$ and whose morphisms are given by natural transformations. More explicitly, we have

Definition 4.1. The loop groupoid $\mathsf{L}[X/G]$ for $[X/G]$ is the action groupoid $[P_G X/G]$ whose space of objects is

$$P_G X := \bigsqcup_{g\in G} P_g X \times \{g\} \qquad \text{with} \qquad P_g X := \{f : [0,1] \to X | f(0)g = f(1)\}$$

and which are endowed with the G-action

$$\begin{array}{rcl} G \times P_G X & \to & P_G X \\ ((f,k), g) & \mapsto & (fg, g^{-1}kg). \end{array}$$

In theorem 2.3 of [18] it is proven that for G a finite group there exist a natural weak homotopy equivalence

$$\mathcal{L}(EG \times_G X) \to EG \times_G P_G X$$

between the free loop space of the homotopy quotient $EG \times_G X$ and the homotopy quotient of the loop groupoid. This proof can be easily generalized to the case that G is discrete.

In this section we will consider $P_G X$ as a G-space and will not work with its homotopy quotient. Now we will give a cosimplicial description for the spaces $P_g X$.

4.2. **Cosimplicial description for $P_g X$.** Let us start by recalling a cosimplicial construction of the space of paths of a topological space X that was done in [14] (we will use the properties of simplicial and cosimplicial spaces that are developed in [25] and in [2] respectively). Take the category Δ whose objects are the finite ordered sets $\mathfrak{n} = \{0, 1, \ldots, n\}$ and whose morphisms $\Delta(\mathfrak{n}, \mathfrak{m})$ are the order preserving maps $s \colon \mathfrak{n} \to \mathfrak{m}$. The morphisms of Δ are generated by:

- The face maps $\delta_i \in \Delta(\mathfrak{n}-1, \mathfrak{n})$, $0 \le i \le n$; the unique order preserving map whose image does not contain i.
- The degeneracy maps $\sigma_i \in \Delta(\mathfrak{n}+1, \mathfrak{n})$, $0 \le i \le n$; the unique surjective order preserving map which repeats i.

These generators satisfy the usual cosimplicial relations. We define a simplicial object in a category $\mathfrak{C}$ to be a contravariant functor $\Delta \to \mathfrak{C}$ and a cosimplicial object in $\mathfrak{C}$ to be a covariant functor $\Delta \to \mathfrak{C}$.

Now let us define the simplicial sets λ^n where $\lambda^n(\mathfrak{m}) := \Delta(\mathfrak{m}, \mathfrak{n})$ with the natural coface and codegeneracy maps induced by Δ. The geometric realization $|\lambda^n|$ of the simplicial set λ^1 is homeomorphic to the n-simplex Δ_n. In particular when $n = 1$, the simplicial space λ^1 has as geometric realization the 1-simplex Δ_1. One could think of it as $\lambda^1(\mathfrak{n}) = \mathfrak{n} + 2$ with cofaces $\lambda^1(\mathfrak{n}) \to \lambda^1(\mathfrak{n}-1)$ and codegeneracies $\lambda^1(\mathfrak{n}-1) \to \lambda^1(\mathfrak{n})$ given by order preserving maps that send 0 to 0, and $n+2$ to $n+1$ in the former case and $n+1$ to $n+2$ in the latter.

Consider the cosimplicial space $\mathbb{P}X := X^{\lambda^1}$ defined by taking the maps from the simplicial set λ^1 to X. Then we have that

$$\mathbb{P}X(\mathfrak{n}) := \mathrm{Map}(\lambda^1(\mathfrak{n}), X) \cong X^{n+2}$$

and the cosimplicial structure maps are the ones induced by the simplicial structure of λ^1. We have the tautology [2, Prop 5.1]

Lemma 4.2. *There is a natural homeomorphism between the space of paths*

$$PX := Map(|\lambda^1|, X) = Map(\Delta_1, X)$$

and the total space $|\mathbb{P}X|$ of the cosimplicial space $\mathbb{P}X$.

Proof. First of all let us describe the cosimplicial structure of $\mathbb{P}X$. We have that $\mathbb{P}X(\mathfrak{n}) = X^{n+2}$ and the cofaces and codegeneracy maps are given by the formulas

$$\begin{aligned} \delta_i(x_0, \dots, x_{n+1}) &= (x_0, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_{n+1}) \quad 0 \le i \le n+1 \\ \sigma_i(x_0, \dots, x_{n+1}) &= (x_0, \dots, x_i, x_{i+2}, \dots, x_{n+1}) \quad 0 \le i \le n-1 \end{aligned}$$

Let Δ_k be the standard k-simplex

$$\Delta_k = \{(t_1, \dots, t_k) \colon 0 \le t_1 \le \cdots \le t_k \le 1\}$$

and consider the maps

$$\begin{aligned} f_k \colon \Delta_k \times PX &\to X^{k+2} \\ (t_1, \dots, t_k) \times \gamma &\mapsto (\gamma(0), \gamma(t_1), \dots, \gamma(t_k), \gamma(1)). \end{aligned} \tag{4.1}$$

Let $\hat{f}_k \colon PX \to \mathrm{Map}(\Delta_k, X^{k+2})$ be the adjoint of f_k and

$$f \colon PX \to \prod_{k \ge 0} \mathrm{Map}(\Delta_k, X^{k+2})$$

the product of the maps $\hat{f}_k$. Then f is a homeomorphism into its image, and moreover the image consists of sequences of maps which commute with the face and degeneracy maps, which is by definition the total space of $\mathbb{P}X$. Therefore we have the homeomorphism

$$f : PX \to \mathrm{im}(f) = |\mathbb{P}X|. \tag{4.2}$$

□

Consider $\overline{\mathbb{P}}_g X$ the subcosimplicial space of $\mathbb{P}X$ defined by the spaces

$$\overline{\mathbb{P}}_g X(\mathfrak{n}) := \{(x_0, \dots, x_{n+1}) \in \mathbb{P}X(\mathfrak{n})\colon x_0 g = x_{n+1}\} \cong X^{n+1}.$$

Clearly the coface and codegeneracy maps are well defined in $\overline{\mathbb{P}}_g X$ and it follows from the homeomorphism (4.2) that the total space of $\overline{\mathbb{P}}_g X$ is homeomorphic to the space of paths $\gamma \in PX$ such that $\gamma(0)g = \gamma(1)$, i.e.

$$f|_{P_g X} : P_g X \to |\overline{\mathbb{P}}_g X| \subset |\mathbb{P}X|$$

is a homeomorphism.

Define the cosimplicial space $\mathbb{P}_g X$ by dropping the last coordinate of $\overline{\mathbb{P}}_g X$ (as the last coordinate x_{n+1} on the n-the level is equal to $x_0 g$) therefore having

$$\mathbb{P}_g X(\mathfrak{n}) := X^{n+1}$$

together with the induced coface and codegeneracy maps given by:

$$\begin{aligned}
\delta_i(x_0, \dots, x_n) &= (x_0, \dots, x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_n) \quad 0 \le i \le n \\
\delta_{n+1}(x_0, \dots, x_n) &= (x_0, \dots, x_n, x_0 g) \\
\sigma_i(x_0, \dots, x_n) &= (x_0, \dots, x_i, x_{i+2}, \dots, x_n) \quad 0 \le i \le n-1.
\end{aligned} \tag{4.3}$$

We can now consider the maps

$$\begin{aligned}
\phi_k \colon \Delta_k \times P_g X &\to X^{k+1} = \mathbb{P}_g X(\mathfrak{n}) \\
(t_1, \dots, t_k) \times \gamma &\mapsto (\gamma(0), \gamma(t_1), \dots, \gamma(t_k)).
\end{aligned} \tag{4.4}$$

together with $\hat{\phi}_k : P_g X \to \mathrm{Map}(\Delta_k, \mathbb{P}_g X)$ their adjoints. Let

$$\phi : P_g X \to \prod_{k \ge 0} \mathrm{Map}(\Delta_k, X^{k+1})$$

denote the product of the maps $\hat{\phi}_k$, then we have

Lemma 4.3. *The image of the map ϕ is the total space of $\mathbb{P}_g X$ and therefore $\phi : P_g X \to |\mathbb{P}_g X|$ is a homeomorphism.*

4.3. **Cochains on loops of $[X/G]$.** Let us denote by

$$C^* := C^*(X; \mathbb{Z})$$

the dg-ring of $\mathbb{Z}$-valued singular cochains on X. Because G acts on X then C^* is endowed with the structure of a $\mathbb{Z}G$-module-dg-ring.

For $g \in G$ let C^*_g be the submodule of $C^* \# G$ generated by the elements of the form $x \otimes g$ as it was defined in (3.5). Note that C^*_g inherits the structure of a C^{*e}-module when one takes $C^{*e} \subset C^* \# G^e$ and $C^*_g \subset C^* \# G$ in the following way: let $a_0 \otimes b_0 \in C^{*e}$ and $x \otimes g \in C^*_g$, then

$$\begin{aligned}
(a_0 \otimes a_1) \cdot x \otimes g &= (a_0 \otimes 1 | a_1 \otimes 1)(x \otimes g) \\
&= (a_0 \otimes 1)(x \otimes g)(a_1 \otimes 1) \\
&= a_0 x g(a_1) \otimes g.
\end{aligned} \tag{4.5}$$

Theorem 4.4. *For X a connected and simply connected CW-complex of finite type, there exists a homomorphism of complexes*

$$B(C^*) \otimes_{C^{*e}} C_g^* \xrightarrow{\simeq} C^*(P_gX;\mathbb{Z})$$

that moreover is a quasi-isomorphism, and therefore induces an isomorphism

$$\mathrm{Tor}^*_{C^{*e}}(C^*, C_g^*) \xrightarrow{\cong} H^*(P_gX;\mathbb{Z}).$$

Proof. Take the cosimplicial space $\mathbb{P}_gX$ an apply to it the singular cochains functor; we obtain the simplicial cochain complex $C^*(\mathbb{P}_gX)$. By Lemma 7.1 of [2] we have that the total complex (or its realization) $|C^*(\mathbb{P}_gX)|$ of the simplicial cochain complex $C^*(\mathbb{P}_gX)$ is quasi-isomorphic to $C^*(P_gX)$, as we have that P_gX is homeomorphic to the total space of $\mathbb{P}_gX$. Then we notice that the total complex $|C^*(\mathbb{P}_gX)|$ is quasi-isomorphic to the complex $B(C^*) \otimes_{C^{*e}} C^*$ when one applies carefully the Eilenberg-Zilber theorem for product spaces [7]. These two facts together imply the theorem. Let us see each one with more detail.

Lemma 4.5. *There is a homomorphism of graded complexes*

$$|C^*(\mathbb{P}_gX)| \longrightarrow C^*(P_gX)$$

that when X is a connected and simply connected CW-complex of finite type it becomes a quasi-isomorphism.

Proof. The lemma is a direct consequence of proposition 5.3 and lemma 7.1 of [2]. Let us see how the homomorphism is defined.

Recall from section 5 of [2] that the group of homogeneous elements of degree r of the total complex $|C^*(\mathbb{P}_gX)|$ is the direct sum of the groups

$$C^{n+r}(\mathbb{P}_gX(\mathfrak{n})) = C^{n+r}(X^{n+1}) = C^*(X^{n+1})[n]^r \qquad \text{for} \quad n \geq 0$$

and therefore

$$|C^*(\mathbb{P}_gX)| = \bigoplus_{n\geq 0} C^*(X^{n+1})[n].$$

Let us consider the composition of the homomorphisms

$$C^{n+r}(X^{n+1}) \xrightarrow{\phi_n^*} C^{n+r}(\Delta_n \times P_gX) \xrightarrow{\int_{\Delta_n}} C^r(P_gX)$$

where the functions ϕ_n were defined in (4.4) and $\int_{\Delta_n}$ evaluates a cochain on the class $[\Delta_n]$, that is: the composition of the Eilenberg-Zilber map together with the evaluation on the class $[\Delta_n]$

$$C^{n+r}(\Delta_n \times P_gX) \longrightarrow C^n(\Delta_n) \otimes C^r(P_gX) \longrightarrow C^r(P_gX).$$

Because the operators $\int_{\Delta_n}$ satisfy the property

$$(-1)^n d\left(\int_{\Delta_n} \alpha\right) = \int_{\Delta_n} d\alpha - \int_{\partial\Delta_n} \alpha$$

we have that the maps

$$C^*(X^{n+1})[n]^r = C^{n+r}(X^{n+1}) \longrightarrow C^r(P_gX)$$

assemble to define homomorphism of complexes of degree zero

$$F : |C^*(\mathbb{P}_g X)| \longrightarrow C^*(P_g X).$$

It follows from proposition 5.3 of [2] that F is a quasi-isomorphism as the connectivity of X (≥ 2) is higher than the connectivity of the simplicial set that defines $\mathbb{P}_g X$ (which in this case is the circle). □

Lemma 4.6. *There is a homomorphism of graded complexes*

$$B(C^*) \otimes_{C^{*e}} C^*_g \to |C^*(\mathbb{P}_g X)|$$

that is moreover a quasi-isomorphism.

Proof. Take the isomorphisms

$$\begin{array}{rcl} C^{*n+1} & \stackrel{\cong}{\to} & C^{*n+2} \otimes_{C^{*e}} C^*_g \\ (a_0|\dots|a_n) & \mapsto & (a_0|\dots|a_n|1) \otimes_{C^{*e}} (1) \end{array}$$

together with the induced differential
(4.6)

$$(a_0|\dots|a_n) \mapsto \sum_{j=0}^{n-1} (-1)^j (a_0|\dots|a_j a_{j+1}|\dots|a_n) + (-1)^n (g(a_n)a_0|\dots|a_{n-1})$$

that comes from the bar differential on $\oplus_{k\geq 0} C^{*k+2}$ (see Appendix A in section 9); note that the last expression in the formula (4.6) comes from the equivalences

$$\begin{array}{rcl} (a_0|\dots|a_n) \otimes_{C^{*e}} (1) & = & (a_0|\dots|a_{n-1}|1) \otimes_{C^{*e}} g(a_n) \\ & = & (g(a_n)a_0|\dots|a_{n-1}|1) \otimes_{C^{*e}} (1). \end{array}$$

Now let us consider the quasi-isomorphisms defined by the Eilenberg-Zilber map

$$C^{*n+1} \stackrel{\simeq}{\to} C^*(X^{n+1}) = C^* \mathbb{P}_g X(\mathfrak{n})$$

and note that the induced differential $C^{*n+1} \to C^{*n}$ coming from the coface maps defined in (4.3) is the same as in (4.6).

We can conclude that the compositions of the maps

$$C^{*n+2} \otimes_{C^{*e}} C^*_g \stackrel{\cong}{\to} C^{*n+1} \stackrel{\simeq}{\to} C^*(X^{n+1})$$

induce the maps

$$C^{*n+2}[n] \otimes_{C^{*e}} C^*_g \stackrel{\cong}{\to} C^{*n+1}[n] \stackrel{\simeq}{\to} C^*(X^{n+1})[n]$$

that induce the desired quasi-isomorphism

$$B(C^*) \otimes_{C^{*e}} C^*_g \stackrel{\simeq}{\to} |C^*(\mathbb{P}_g X)|.$$

□

Now we can finish with the proof of theorem 4.4 by composing the quasi-isomorphisms defined in the lemmas 4.5 and 4.6

$$B(C^*) \otimes_{C^{*e}} C^*_g \xrightarrow{\simeq} |C^*(\mathbb{P}_g X)| \xrightarrow{\simeq} C^*(P_g X)$$

that moreover induce the isomorphisms

$$\mathrm{Tor}_{C^{*e}}(C^*, C^*_g) \xrightarrow{\cong} H^*(|C^*(\mathbb{P}_g X)|) \xrightarrow{\cong} H^*(P_g X).$$

□

We can assemble all the C^*_g's into $C^*\#G$ thus obtaining

Corollary 4.7. *If X is connected and simply connected and G is finite, then there is a quasi-isomorphism*

$$B(C^*X) \otimes_{C^*X^e} C^*X\#G \xrightarrow{\simeq} C^*(P_G X)$$

that induces the isomorphism

$$\mathrm{Tor}_{C^*X^e}(C^*X, C^*X\#G) \cong H^*(P_G X).$$

The previous corollary together with theorem 3.2 implies that

Proposition 4.8. *If X is connected and simply connected and G is finite, then there is an isomorphism of graded groups*

$$HH_*(C^*X\#G, C^*X\#G) \cong \mathrm{Tor}_{\mathbb{Z}G}(\mathbb{Z}, C^*(P_G X)),$$

and whenever G acts freely on X the isomorphism becomes

$$HH_*(C^*X\#G, C^*X\#G) \cong H^*\left(C^*(P_G X)^G\right).$$

4.4. **Chains on loops of $[X/G]$.** From the previous section we have the quasi-isomorphisms

$$B(C^*) \otimes_{C^{*e}} C^*_g \xrightarrow{\cong} \bigoplus_{n\geq 0} C^{*n+1}[n] \xrightarrow{\simeq} |C(\mathbb{P}_g X)| \xrightarrow{\simeq} C^*(P_g X). \tag{4.7}$$

If $P_g X$ is of finite type i.e. the cohomology is finitely generated in each degree, we can dualize the maps of (4.7) thus obtaining quasi-isomorphisms

$$C_*(P_g X) \xrightarrow{\simeq} |C(\mathbb{P}_g X)|^\vee \xrightarrow{\simeq} \bigoplus_{n\geq 0} C^{*n+1}[n]^\vee \overset{\simeq}{\cong} \bigoplus_{n\geq 0} \mathcal{H}om_{\mathbb{Z}}(C^{*n}[n], C^{*\vee}_g) \tag{4.8}$$

where $T^\vee := \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Z})$ and the last map is induced by the isomorphisms $\mathcal{H}om_{\mathbb{Z}}(C^{*n}, C^{*\vee}_g) \to C^{*n+1\vee}$ with $\phi \mapsto \bar{\phi}$ and

$$\bar{\phi}(a_1|\ldots|a_n|b) = \phi(a_1|\ldots|a_n)(b).$$

Whenever X is a compact closed oriented manifold of dimension l, there is a homomorphism of graded groups

$$C^*X \to C_{l-*}X$$

that induces an isomorphism in cohomology, and that moreover induces an structure of C^*X^e-module on $C_{l-*}X$. Therefore we could apply the natural isomorphism

$$\mathcal{H}om_{\mathbb{Z}}(C^{*n}, C_g^{*\vee}) \cong \mathcal{H}om_{C^{*e}}(C^{*n+2}, C_g^{*\vee})$$

that yields then that there is a quasi-isomorphism

$$C_*(P_gX) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C_g^{*\vee})$$

that induces the isomorphism

$$H_*(P_gX) \cong H^*\mathcal{H}om_{C^{*e}}(B(C^*), C_g^{*\vee}).$$

The previous proof is very sketchy and not rigorous at all; we postpone its formal proof to the next section.

In the next section we will show explicitly how the singular chains of P_gX are related to the complex $\mathcal{H}om_{C^{*e}}(B(C^*), C_g^*)$, and moreover we will show how the ring structure of $H_*(P_GX)$ defined in [18] is isomorphic to the ring $H^*\mathcal{H}om_{C^{*e}}(C^*, C^*\#G)$ whenever G is finite and X is a compact, connected and simply connected oriented manifold.

5. String topology for orbifolds

This section is devoted to the construction of the topological counterpart for the G-module dg-ring

$$\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$$

whenever we have a global quotient orbifold $[M/G]$ with M a differentiable, oriented and compact manifold, G finite group acting by orientation preserving diffeomorphisms and $C^* = C^*(M, \mathbb{Z})$.

In section 5.1 we will start by recalling the construction of the orbifold string topology of $[M/G]$ performed in [18], which is based on the union of spectra

$$P_GM^{-TM} := \bigsqcup_{g\in G} P_gM^{-TM}$$

and the maps

$$P_gM^{-TM} \wedge P_hM^{-TM} \to P_{gh}M^{-TM}. \tag{5.1}$$

Then in section 5.2 we will construct for each g a cosimplicial spectrum $\mathfrak{P}_gM$ such that its total spectrum is homeomorphic to P_gM^{-TM}. With the cosimplicial spectrum $\mathfrak{P}_gM$ in hand, we will construct a quasi-isomorphism between $C_*(P_gM^{-TM})$ and $\mathcal{H}om_{C^{*e}}(B(C^*), C_g^*)$ which will assemble into a quasi-isomorphism

$$C^*(P_GM^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G).$$

In section 5.3 we will focus on multiplicative issues. We will construct maps of cosimplicial spectra

$$\mathfrak{P}_gM \wedge \mathfrak{P}_hM \to \mathfrak{P}_{gh}M$$

that will be compatible with the map of (5.1) and we will show that after passing to chains, it will be compatible with the ring structure of

$$\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G).$$

We will show that the quasi-isomorphism

$$C^*(P_G M^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$$

will be a G-equivariant map of A_∞ rings, and in particular we will be able show that there is G-equivariant isomorphism of rings

$$H_*(P_G M^{-TM}) \cong H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G).$$

5.1. **Orbifold string topology.** This section is based on the construction done in [18] of the orbifold loop spectra $P_G M^{-TM}$.

Take an embedding $\rho : M \to V^k$ where V is a k-dimensional real representation of G and ρ is G-equivariant (for its existence see Appendix B of [12] or [21]). Let $\nu \to M$ the normal bundle of the embedding and denote by ν_+ its Thom construction. Define the spectrum

$$M^{-TM} := \Sigma^{-k}\nu_+$$

as the k-th de-suspension of the Thom space ν_+ and note that it is moreover a naive G-spectrum (G acts on the pointed space ν_+). Recall also that by S-duality the diagonal map $\Delta : M \to M \times M$ induces a map of spectra

$$\Delta^* : M^{-TM} \wedge M^{-TM} \to M^{-TM}$$

which makes M^{-TM} into a ring spectrum. Passing to chains, the map

$$\Delta^* : C_*(M^{-TM}) \otimes C_*(M^{-TM}) \to C_*(M^{-TM})$$

is with respect to the duality identification

$$C_{-*}(M^{-TM}) \to C^*M$$

the cup product on cochains

$$\Delta^* : C^*M \otimes C^*M \to C^*M.$$

In our case, as the embedding $M \to V$ is equivariant, the previous maps and identifications are G equivariant.

Let us now consider the twisted diagonal

$$\Delta_g : M \to M \times M \qquad m \mapsto (m, mg)$$

and denote by $N^g \to \Delta_g M$ the normal bundle of the submanifold $\Delta_g M$ defined by the embedding. We have then that $T\Delta_g M \oplus N_g = T(M \times M)|_{\Delta_g M}$ and therefore we have an induced map of spectra

$$\Delta_g^* : M^{-TM} \wedge M^{-TM} \to M^{\Delta_g^* N_g \ominus TM \ominus \Delta_g^* N_g} = M^{-TM}$$

which after passing to chains, dualizes the map

$$\Delta_g^* : C^*M \otimes C^*M \to C^*M \qquad a \otimes b \mapsto a \cdot g(b).$$

Now, for $t \in \mathbb{R}$ let $e_t : P_g M \to M$ be the evaluation of a map at the time t: $e_t(f) := f(t)$. Consider the space of composable maps

$$P_g M \,{}_1{\times}_0\, P_h M = \{(\phi, \psi) \in P_g M \times P_h M | \phi(1) = \psi(0)\}$$

and note that they fit into the pullback square

$$\begin{tikzcd} P_g M \,{}_1{\times}_0\, P_h M \arrow[r] \arrow[d, "e_0 \circ \pi_1"] & P_g M \times P_h M \arrow[d, "e_0 \times e_0"] \\ M \arrow[r, "\Delta_g"] & M \times M \end{tikzcd}$$

where the downward arrow $e_0 \circ \pi_1$ is the evaluation at zero of the first map.

The normal bundle of the upper horizontal map becomes isomorphic to the pullback under $e_0 \circ \pi_1$ of the normal bundle of the map Δ_g and therefore we can construct the Thom-Pontrjagin collapse maps that makes the diagram commutative

$$\begin{tikzcd} P_g M \times P_h M \arrow[r] \arrow[d, "e_0 \times e_0"] & P_g M \,{}_1{\times}_0\, P_h M^{\pi^* e_0^* N_g} \arrow[d, "e_0 \circ \pi_1"] \\ M \times M \arrow[r] & M^{N_g} \end{tikzcd}$$

and by inverting the tangent bundle of $M \times M$ on the lower left hand we obtain the commutative square

$$\begin{tikzcd} P_g M^{-e_0^* TM} \wedge P_h M^{-e_0^* TM} \arrow[r] \arrow[d, "e_0 \times e_0"] & P_g M \,{}_1{\times}_0\, P_h M^{-\pi^* e_0^* TM} \arrow[d, "e_0 \circ \pi_1"] \\ M^{-TM} \wedge M^{-TM} \arrow[r, "\Delta_g^*"] & M^{-TM}. \end{tikzcd}$$

The concatenation of paths in $P_g M \,{}_1{\times}_0\, P_h M$ produces a map

$$P_g M \,{}_1{\times}_0\, P_h M \to P_{gh} M$$

which induces the map of spectra

$$P_g M \,{}_1{\times}_0\, P_h M^{-\pi^* e_0^* TM} \to P_{gh} M^{-e_0^* TM}$$

that composed with the upper horizontal map of diagram defines the map

$$\mu_{g,h} : P_g M^{-e_0^* TM} \wedge P_h M^{-e_0^* TM} \to P_{gh} M^{-e_0^* TM}$$

which fits into the commutative diagram of spectra

$$\begin{tikzcd} P_g M^{-e_0^* TM} \wedge P_h M^{-e_0^* TM} \arrow[r, "\mu_{g,h}"] \arrow[d, "e_0 \times e_0"] & P_{gh} M^{-e_0^* TM} \arrow[d, "e_0"] \\ M^{-TM} \wedge M^{-TM} \arrow[r, "\Delta_g^*"] & M^{-TM}. \end{tikzcd}$$

If we denote the spectrum $P_g M^{-TM}$ as

$$P_g M^{-TM} := P_g M^{-e_0^* TM},$$

we define

$$P_G M^{-TM} := \bigsqcup_{g\in G} P_g M^{-TM}$$

which by assembling the maps $\mu_{g,h}$ we define a map

$$\mu : P_G M^{-TM} \wedge P_G M^{-TM} \to P_G M^{-TM}. \tag{5.2}$$

where we have denoted

$$P_G M^{-TM} \wedge P_G M^{-TM} := \bigsqcup_{(g,h)\in G\times G} P_g M^{-TM} \wedge P_h M^{-TM}.$$

Note that the compositions $\mu \circ (\mu \times 1)$ and $\mu \circ (1 \times \mu)$ do not agree, but they are homotopically equivalent.

Recall from definition 4.1 that $P_G M$ is a G-space where for $k \in G$ and $f \in P_g M$ we have that $fk \in P_{k^{-1}gk}M$. Because the embedding of M to the representation $\rho : M \to V$ is G-equivariant, the spectra M^{-TM} acquires an action of G, and moreover, for $k \in G$ there is an induced map $P_g M^{-TM} \to P_{k^{-1}gk}M^{-TM}$ of spectra, where the based point in $P_g M^{-TM}$ is mapped to the based point in $P_{k^{-1}gk}M^{-TM}$.

It follows that for $k \in G$ we have the commutative diagram

$$\begin{tikzcd}
P_g M^{-TM} \wedge P_h M^{-TM} \arrow[r, "\mu_{g,h}"] \arrow[d, "k\times k"] & P_{gh} M^{-TM} \arrow[d, "k"] \\
P_{k^{-1}gk} M^{-e_0^*TM} \wedge P_{k^{-1}hk} M^{-TM} \arrow[r, "\mu_{k^{-1}gk,k^{-1}hk}"] & P_{k^{-1}ghk} M^{-TM}
\end{tikzcd}$$

which implies that the map $\mu : P_G M^{TM} \wedge P_G M^{-TM} \to P_G M^{-TM}$ is G equivariant.

Therefore we could think of $P_G M^{-TM}$ as ring spectrum with a G action and therefore $C_*(P_G M^{-TM})$ becomes a G-module A_∞-ring.

Passing to homology, we get a G-module ring $H_*(P_G M^{-TM}; \mathbb{Z})$ whose product structure was called the G-string product in [18], and the induced ring structure on the invariant set

$$H_*(P_G M^{-TM}; \mathbb{Q})^G$$

was called in [18] the **orbifold string topology ring**.

5.2. **Cosimplicial spectra.** In this section we will describe a cosimplicial model for the spectra $P_g M^{-TM}$ that will allow us to give a natural way of relating the singular chains $C_*(P_g M^{-TM})$ to the complex $\mathcal{H}om_{C^{*e}}(B(C^*), C_g^*)$. This section is a generalization of section 3 of [5] and we will mimic their construction.

Recall from (4.4) the functions ϕ_k

$$\begin{array}{rcl} \phi_k \colon \Delta_k \times P_g X & \to & M^{k+1} = \mathbb{P}_g M(\mathfrak{n}) \\ (t_1, \ldots, t_k) \times \gamma & \mapsto & (\gamma(0), \gamma(t_1), \ldots, \gamma(t_k)). \end{array}$$

and consider the commutative diagram

$$\begin{tikzcd} \Delta_k \times P_g M \arrow[r, "\phi_k"] \arrow[d, "e"] & M^{k+1} \arrow[d, "\pi_1"] \\ M \arrow[r, "="] & M \end{tikzcd}$$

where $e((t_0, t_1, \dots, t_k), \gamma) \mapsto \gamma(0)$ and the right hand vertical map is the projection on the first coordinate. Pulling back the virtual bundle $-TM$ under e and π_1 we get a map of Thom spectra, that by abuse of notation we still call ϕ_k

$$\phi_k : (\Delta_k)_+ \wedge P_g M^{-TM} \longrightarrow M^{-TM} \wedge (M^k)_+.$$

Taking adjoints, we get a map of spectra

$$\phi : P_g M^{-TM} \longrightarrow \prod_k \mathrm{Map}((\Delta_k)_+, M^{-TM} \wedge (M^k)_+)$$

that is just the induced map of Thom spectra of the map

$$\phi : P_g M \to \prod_{k \geq 0} \mathrm{Map}(\Delta_k, M^{k+1})$$

described in lemma 4.3.

We are now ready to define the cosimplicial spectrum $\mathfrak{P}_g M$; it will be the cosimplicial Thom spectrum of the cosimplicial virtual bundle $-TM$ on $\mathbb{P}_g M$. More explicitly, let $\mathfrak{P}_g M$ be the cosimplicial spectrum whose k-simplices are the spectrum

$$\mathfrak{P}_g M_k := M^{-TM} \wedge (M^k)_+$$

and whose coface and codegeneracy maps are:

$$\begin{aligned} \delta_0(u; x_1, \dots, x_{k-1}) &= (u; y, x_1, \dots, x_{k-1}) \\ \delta_i(u; x_1, \dots, x_{k-1}) &= (u; x_1, \dots x_{i-1}, x_i, x_i, x_{i+1}, \dots, x_{k-1}), \quad 1 \leq i \leq k-1 \\ \delta_k(u; x_1, \dots, x_{k-1}) &= (v; x_1, \dots, x_{k-1}, z) \\ \sigma_i(u; x_1, \dots, x_{k+1}) &= (u; x_1, \dots, x_i, x_{i+2}, \dots, x_{k+1}) \quad 0 \leq i \leq k \end{aligned}$$

with $\mu(u) = (u, y)$ and $\nu(u) = (z, v)$ where

$$\mu : M^{-TM} \to M^{-TM} \wedge M_+ \qquad \nu : M^{-TM} \to M_+ \wedge M^{-TM}$$

are the maps of Thom spectra induced by the diagonal maps $\Delta : M \to M \times M$ and $\Delta_g : M \to M \times M$ respectively. μ and ν are the maps of Thom spectra induced by the maps of virtual bundles $\Delta_* : -TM \to -\pi_1^* TM$ and $\Delta_{g*} : -TM \to -\pi_2^* TM$.

Let $|\mathfrak{P}_g M|$ be the total spectrum of the cosimplicial spectrum $\mathfrak{P}_g M$ i.e. it consists of sequences of maps $\{\gamma_k\}$ in

$$\prod_k \mathrm{Map}((\Delta_k)_+; M^{-TM} \wedge (M^k)_+)$$

which commute with the coface and codegeneracy maps. Therefore we can conclude that by applying the Thom spectrum functor for the virtual bundle of lemma 4.3 we get

Proposition 5.1. *The map*

$$\phi : P_g M^{-TM} \longrightarrow \prod_k Map((\Delta_k)_+, M^{-TM} \wedge (M^k)_+)$$

induces a homeomorphism between the spectrum $P_g M^{-TM}$ *and the total spectrum of* $\mathfrak{P}_g M$

$$\phi : P_g M^{-TM} \xrightarrow{\cong} |\mathfrak{P}_g M|.$$

As a consequence of lemma 4.5 after passing to Thom spectra, we get that the maps $\phi_k : (\Delta_k)_+ \wedge P_g M^{-TM} \to M^{-TM} \wedge (M^k)_+$ define maps of cochains

$$C^*(M^{-TM} \wedge (M^k)_+)[k] \to C^*(P_g M^{-TM})$$

which assemble to give a map from the total complex of the simplicial cochain complex $|C^*(\mathfrak{P}_g M)|$ to the cochain complex of the total spectrum $|\mathfrak{P}_g M|$; therefore we have

Lemma 5.2. *There is a homomorphism of graded complexes*

$$|C^*(\mathfrak{P}_g M)| \longrightarrow C^*(P_g M^{-TM})$$

that when M *is a connected and simply connected manifold it becomes a quasi-isomorphism.*

If we denote

$$|C^*(\mathfrak{P}_g M)|^\vee := \mathcal{H}om_{\mathbb{Z}}(|C^*(\mathfrak{P}_g M)|, \mathbb{Z})$$

then let us show that

Lemma 5.3. *There is a map of graded complexes*

$$|C^*(\mathfrak{P}_g M)|^\vee \to \mathcal{H}om_{C^{*e}}(B(C^*), C^*_g)$$

that moreover is a quasi-isomorphism.

Proof. The cochains of the k simplices of $\mathfrak{P}_g M$ become chain homotopy equivalent to

$$C^*(M^{-TM} \wedge (M^k)_+) \cong C^*(M)^k \otimes C^*(M^{-TM})$$

and after dualizing we get that

$$\begin{aligned} C^*(M^{-TM} \wedge (M^k)_+)[k]^\vee \cong & \mathcal{H}om_{\mathbb{Z}}(C^*(M)[1]^{\otimes k} \otimes C^*(M^{-TM}), \mathbb{Z}) \\ \cong & \mathcal{H}om_{\mathbb{Z}}(C^*(M)[1]^{\otimes k}, C_*(M^{-TM})) \\ \cong & \mathcal{H}om_{\mathbb{Z}}(C^*(M)[1]^{\otimes k}, C^*(M)) \\ \cong & \mathcal{H}om_{C^{*e}}(C^* \otimes C^*[1]^{\otimes k} \otimes C^*, C^*_g). \end{aligned}$$

Here we have used the fact that Atiyah duality produces an equivalence of spectra, and therefore there exists a map

$$\alpha_* : C_*(M^{-TM}) \to C^*(M)$$

compatible with the $C^*(M)^e$-module structure of $C^*(M)$, that moreover is a homotopy equivalence. This fact was proven in [4] and it is its main result.

Now, the same proof as in lemma 4.6 will tell us that the above isomorphisms are compatible with the differentials, and therefore we have the desired quasi-isomorphism

$$|C^*(\mathfrak{P}_g M)|^\vee \xrightarrow{\cong} \mathcal{H}om_{C^{*e}}(B(C^*), C^*_g).$$

□

Lemma 5.2 together with lemma 5.3 allow us to conclude that

Theorem 5.4. *For M a connected and simply connected compact manifold, there exists a homomorphism of graded complexes*

$$C_*(P_g M^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^*_g)$$

that moreover is a quasi-isomorphism, and therefore induces an isomorphism of graded groups

$$H_*(P_g M^{-TM}) \xrightarrow{\cong} H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*_g).$$

From the previous theorem, assembling the maps for all g in G, we can deduce that

Corollary 5.5. *For M a connected and simply connected compact manifold, there exists a homomorphism of graded complexes*

$$C_*(P_G M^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$$

that moreover is a quasi-isomorphism and G-equivariant, and therefore induces a G-equivariant isomorphism of graded groups

$$H_*(P_G M^{-TM}) \xrightarrow{\cong} H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G).$$

Proof. The only thing left to prove is that the maps $P_g M^{-TM} \to |\mathfrak{P}_g M|$ induce an equivariant map

$$P_G M^{-TM} \to \bigsqcup_{g\in G} |\mathfrak{P}_g M|,$$

but this follows from the fact that the evaluation maps $e_t : P_G M \to M$ are G equivariant. □

5.3. **Multiplicative structures.** In this section we will show how the map of corollary 5.5 is compatible with the multiplicative structure of $C_*(P_G M^{-TM})$ coming from the map μ (5.2), and the natural ring structure on $\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$ that was explained in section 3.3.1. To achieve this, we will endow the cosimplicial spectra $\mathfrak{P}_g M$ with multiplicative maps

$$\tilde{\mu}_{g,h} : \mathfrak{P}_g M \wedge \mathfrak{P}_h M \to \mathfrak{P}_{gh} M$$

that once realized will be compatible with the maps $\mu_{g,h}$ and that after passing to chains, will realize the ring structure of $\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$.

Let us define the maps

$$\begin{aligned} \tilde{\mu}^{k,l}_{g,h} : \left(M^{-TM} \wedge (M^k)_+\right) \wedge \left(M^{-TM} \wedge (M^l)_+\right) &\longrightarrow M^{-TM} \wedge (M^{k+l})_+ \\ (u; x_1, \dots, x_k) \wedge (v; y_1, \dots, y_l) &\mapsto (\Delta^*_g(u,v); x_1, \dots, x_k, y_1, \dots, y_l) \end{aligned}$$

that define maps of the simplices

$$\tilde{\mu}^{k,l}_{g,h} : (\mathfrak{P}_g M)_k \wedge (\mathfrak{P}_h M)_l \to (\mathfrak{P}_{gh} M)_{k+l}$$

which commute with the coface and codegeneray operators, and therefore induce maps of cosimplicial spectra

$$\tilde{\mu}_{g,h} : \mathfrak{P}_g M \wedge \mathfrak{P}_h M \to \mathfrak{P}_{gh} M.$$

Taking the total spectrum we get maps of spectra

$$|\tilde{\mu}_{g,h}| : |\mathfrak{P}_g M| \wedge |\mathfrak{P}_h M| \to |\mathfrak{P}_{gh} M|$$

that induce pairings that are A_∞-associative.

Applying the chains functor to the map $\tilde{\mu}^{k,l}_{g,h}$ we get the map

$$\begin{aligned} (\tilde{\mu}^{k,l}_{g,h})_* : (C_*(M)^k \otimes C_*(M^{-TM})) \otimes (C_*(M)^l \otimes C_*(M^{-TM})) \to \\ C_*(M)^{k+l} \otimes C_*(M^{-TM}) \end{aligned}$$

that by Atiyah duality induces the map

$$\begin{aligned} (\tilde{\mu}^{k,l}_{g,h})_* : (C_*(M)^k \otimes C^*(M)) \otimes (C_*(M)^l \otimes C^*(M)) \to C_*(M)^{k+l} \otimes C^*(M) \\ (a_1 \otimes \cdots \otimes a_k \otimes \alpha) \otimes (b_1 \otimes \cdots b_l \otimes \beta) \mapsto \\ a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l \otimes \Delta^*_g(\alpha \otimes \beta) \\ = a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l \otimes \alpha \cdot g(\beta). \end{aligned}$$

Composing with the natural isomorphisms

$$C_*(M)^k \otimes C^*(M) \cong \mathcal{H}om_{C^*(M)^e}(C^*(M)^{k+2}, C^*(M))$$

wee see that the map $(\tilde{\mu}^{k,l}_{g,h})_*$ induces the multiplicative structure that for

$$\phi_g \in \mathcal{H}om_{C^*(M)^e}(C^*(M)^{k+2}, C^*(M)), \ \ \psi_h \in \mathcal{H}om_{C^*(M)^e}(C^*(M)^{l+2}, C^*(M))$$

defines the function

$$(a_0 | \dots | a_{k+l+1}) \mapsto (-1)^{|\psi_h|\varepsilon_k} \phi_g(a_0 | \dots | a_k | 1) g(\psi_h(1 | a_{k+1} | \dots | a_{k+l+1})$$

which agrees with the definition of the product structure of $\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$ done in section 3.3.1.

We have then

Lemma 5.6. *The maps of lemma 5.3*

$$|C^*(\mathfrak{P}_g M)|^\vee \to \mathcal{H}om_{C^{*e}}(B(C^*), C^*_g)$$

are compatible with the multiplicative structures on $\bigoplus_{g\in G} |C^*(\mathfrak{P}_g M)|^\vee$ and $\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$, which on the first term, the multiplicative structure is induced by the maps $(\tilde{\mu}^{k,l}_{g,h})_*$. In particular we have an isomorphism of rings

$$\bigoplus_{g\in G} H^*(|C^*(\mathfrak{P}_g M)|^\vee) \xrightarrow{\cong} H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G).$$

We are left with providing the relationship between the maps $\mu_{g,h}$ and the maps $\tilde{\mu}_{g,h}$; this will be achieved with the following theorem

Theorem 5.7. *The multiplicative structures that the maps $\mu_{g,h}$ and $\tilde{\mu}_{g,h}$ define are compatible in the sense that the following diagram homotopy commutes:*

$$\begin{tikzcd} P_g M^{-TM} \wedge P_h M^{-TM} \arrow[r, "\mu_{g,h}"] \arrow[d, "\phi\wedge\phi"] & P_{gh} M^{-TM} \arrow[d, "\phi"] \\ |\mathfrak{P}_g M| \wedge |\mathfrak{P}_h M| \arrow[r, "\tilde{\mu}_{g,h}"] & |\mathfrak{P}_{gh} M| \end{tikzcd}$$

where ϕ is the map of proposition 5.1.

Proof. The proof is almost identical to the proof of theorem 13 on [5]. The only difference is that whenever it is used the diagonal map Δ on the proof of theorem 13 of [5] it needs to be replaced by the twisted diagonal Δ_g. We will not reproduce the proof here. □

Lemma 5.6 together with theorem 5.7 imply that the G-equivariant quasi-isomorphism of corollary 5.5 is compatible with the multiplicative structures previously defined, and therefore we can finish this section with

Theorem 5.8. *For M a connected and simply connected compact manifold, there exists a quasi-isomorphism of graded complexes*

$$\Phi : C_*(P_G M^{-TM}) \xrightarrow{\simeq} \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)$$

which is G-equivariant, that furthermore makes the following diagram commute

$$\begin{tikzcd} C_*(P_G M^{-TM}) \otimes C_*(P_G M^{-TM}) \arrow[r, "\mu_*"] \arrow[d, "\phi_*\otimes\phi_*"] & C_*(P_G M^{-TM}) \arrow[d, "\phi_*"] \\ \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G) \otimes \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G) \arrow[r, "\cdot"] & \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G). \end{tikzcd}$$

Hence, Φ induces a G-equivariant isomorphism of rings

$$\Phi : H_*(P_G M^{-TM}) \xrightarrow{\cong} H^* \mathcal{H}om_{C^{*e}}(B(C^*), C^* \# G).$$

The previous theorem 5.8, together with theorem 3.2 and the results of the section 10 imply that

Theorem 5.9. *For M a connected and simply connected and G a finite group acting on M, then there exits an isomorphism of graded rings*

$$HH^*(C^*M\#G, C^*M\#G) \cong \operatorname{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, C_*(P_G M^{-TM})).$$

6. Homotopical realization for $HH^*(C^*M\#G, C^*M\#G)$

From theorem 5.9 we know that have an isomorphism of rings

$$HH^*(C^*M\#G, C^*M\#G) \cong \operatorname{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, C_*(P_G M^{-TM}))$$

which can be further extended to an isomorphism of rings

$$HH^*(C^*M\#G, C^*M\#G) \cong H^* \left(C^*EG \otimes C_*(P_G M^{-TM})\right)^G \tag{6.1}$$

as we know that

$$\begin{aligned} \operatorname{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, C_*(P_G M^{-TM})) &\cong H^* \mathcal{H}om_{\mathbb{Z}G}(C_*EG, C_*(P_G M^{-TM})) \\ &\cong H^* \mathcal{H}om_{\mathbb{Z}G}(\mathbb{Z}, C^*EG \otimes C_*(P_G M^{-TM})) \\ &= H^* \left(C^*EG \otimes C_*(P_G M^{-TM})\right)^G . \end{aligned} \tag{6.2}$$

We would like to have a topological construction associated to the Hochschild cohomology of $C^*M\#G$. One deterrent for its existence comes from the fact that on the right hand side of (6.1) we have a mixture of chain with cochain complexes.

To overcome this problem we will construct a pro ring spectrum EG^{-TEG} associated to EG, that will allow us to change the cochain complex C^*EG by the chain complex of EG^{-TEG}. With this idea in mind, let us generalize the construction of the string topology of BG that can be found in [10, 11] to the orbifold case.

Let $EG_1 \subset \cdots \subset EG_n \subset EG_{n+1} \subset \cdots EG$ be a finite dimensional manifold approximation of the universal G-principal bundle $EG \to BG$. Consider the maps

$$P_G M \times_G EG_n \stackrel{e_0}{\to} M \times_G EG_n$$

where by abuse of notation we denote the evaluation at the time t of a pair $(f, \lambda) \in P_G M \times_G EG_n$ also by e_t.

Take the Thom spectra

$$(P_G M \times_G EG_n)^{-e_0^* TM_n}$$

and let us show that indeed it is a ring spectra. Consider the diagram

$$\begin{tikzcd}
(P_GM\ {}_1\times_0\ P_GM)\times_G EG_n \arrow[r, "\widetilde{\Delta}"] \arrow[d, "e_0"] & (P_GM\times_G EG_n)\times(P_GM\times_G EG_n) \arrow[d, "e_0\times e_0"] \\
M\times_G EG_n \arrow[r, "\Delta"] & (M\times_G EG_n)\times(M\times_G EG_n)
\end{tikzcd}$$

where Δ is the diagonal inclusion and $P_GM\ {}_1\times_0\ P_GM$ fits in the pullback square

$$\begin{tikzcd}
P_GM\ {}_1\times_0\ P_GM \arrow[r] \arrow[d, "e_\infty"] & P_GM\times P_GM \arrow[d, "e_1\times e_0"] \\
M \arrow[r, "\mathrm{diag}"] & M\times M.
\end{tikzcd}$$

As the normal bundle of the inclusion $P_GM\ {}_1\times_0\ P_GM\to P_GM\times P_GM$ is isomorphic to e_∞^*TM, then the normal bundle of the inclusion

$$\widetilde{\Delta}:(P_GM\ {}_1\times_0\ P_GM)\times_G EG_n\to(P_GM\times_G EG_n)\times(P_GM\times_G EG_n)$$

is isomorphic to $e_\infty^*TM_n$. Then by the Thom-Pontryagin construction we have a map

$$(P_GM\times_G EG_n)\times(P_GM\times_G EG_n)\to((P_GM\ {}_1\times_0\ P_GM)\times_G EG_n)^{e_\infty^*TM_n}$$

which induces a map of spectra

$$\nu:(P_GM\times_G EG_n)^{-e_1^*TM_n}\wedge(P_GM\times_G EG_n)^{-e_0^*TM_n}\longrightarrow((P_GM\ {}_1\times_0 P_GM)\times_G EG_n)^{-e_\infty^*TM_n}.$$

Let us recall the concatenation map

$$\begin{array}{rcl}
\mu: P_GM\ {}_0\times_1\ P_GM & \to & P_GM \\
((\phi,g),(\psi,h)) & \mapsto & (\phi\circ\psi, gh)
\end{array}$$

where

$$\phi\circ\psi(t):=\begin{cases}\phi(2t) & \text{for } \ 0\le t<\frac{1}{2}\\ \psi(2t-1) & \text{for } \ \frac{1}{2}\le t\le 1;\end{cases}$$

We have the commutative square

$$\begin{tikzcd}
(P_GM\ {}_0\times_1\ P_GM)\times_G EG_n \arrow[r, "\mu"] \arrow[d, "e_\infty"] & P_GM\times_G EG_n \arrow[d, "e_{\frac{1}{2}}"] \\
M\times_G EG_n \arrow[r, "="] & M\times_G EG_n
\end{tikzcd}$$

that induces a map on spectra

$$((P_GM_0\times_1 P_GM)\times_G EG_n)^{-e_\infty^*T(M\times_G EG_n)}\xrightarrow{\overline{\mu}}(P_GM\times_G EG_n)^{-e_{\frac{1}{2}}^*T(M\times_G EG_n)}$$

which composed with the map ν gives us a map of spectra

$$(P_GM\times_G EG_n)^{-e_1^*T(M\times_G EG_n)}\wedge(P_GM\times_G EG_n)^{-e_0^*T(M\times_G EG_n)}$$
$$\longrightarrow(P_GM\times_G EG_n)^{-e_{\frac{1}{2}}^*T(M\times_G EG_n)}.$$

Because all the maps $e_0, e_{\frac{1}{2}}, e_1$ are homotopy equivalent, then the bundles $e_0^*TM_n \cong e_{\frac{1}{2}}^*TM_n \cong e_1^*TM_n$ are all isomorphic. Therefore we can construct a map of spectra

$$(P_GM \times_G EG_n)^{-e_0^*T(M\times_G EG_n)} \wedge (P_GM \times_G EG_n)^{-e_0^*T(M\times_G EG_n)} \longrightarrow (P_GM \times_G EG_n)^{-e_0^*T(M\times_G EG_n)}.$$

that makes $(P_GM \times_G EG_n)^{-e_0^*TM_n}$ into a ring spectra.

The inclusions

$$\begin{array}{ccccccc}
\cdots \longrightarrow & P_GM \times_G EG_n & \longrightarrow & P_GM \times_G EG_{n+1} & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & \\
\cdots \longrightarrow & M \times_G EG_n & \longrightarrow & M \times_G EG_{n+1} & \longrightarrow \cdots
\end{array}$$

induce maps of ring spectra

(6.3)
$$(P_GM \times_G EG_n)^{-e_0^*T(M\times_G EG_n)} \overset{\rho_n^{n+1}}{\longleftarrow} (P_GM \times_G EG_{n+1})^{-e_0^*T(M\times_G EG_{n+1})}.$$

Definition 6.1. The previous system of ring spectra we will denote it by

$$\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)},$$

and we will call it **the free loop space pro ring spectrum associated to the orbifold**.

Note that the maps of ring spectra in (6.3) do not assemble into an inverse system of ring spectra. The reason for this is that each of the maps ρ_n^{n+1} of ring spectra in (6.3) depends explicitly on an equivariant embedding $EG_{n+1} \to V_{n+1}$ of EG_{n+1} into a representation of G. Therefore the composition $\rho_{n-1}^n \circ \rho_n^{n+1}$ does not necessarily agree with the map of spectra

$$(P_GM \times_G EG_{n-1})^{-e_0^*T(M\times_G EG_{n-1})} \overset{\rho_{n-1}^{n+1}}{\longleftarrow} (P_GM \times_G EG_{n+1})^{-e_0^*T(M\times_G EG_{n+1})}$$

induced by the inclusion $EG_{n-1} \subset EG_{n+1}$.

Nevertheless, the maps ρ_{n-1}^{n+1} and $\rho_{n-1}^n \circ \rho_n^{n+1}$ are homotopically equivalent, and therefore after applying the homology functor we indeed get an inverse system of graded rings. Therefore we define

Definition 6.2. The homology of the free loop space pro ring spectrum associated to the orbifold $[M/G]$ is

$$H_*^{pro}(\mathcal{L}(M \times_G EG)^{-T(M\times_G EG)}) := \lim_{\overleftarrow{n}} H_*((P_GM \times_G EG_n)^{-e_0^*T(M\times_G EG_n)})$$

We claim

Theorem 6.3. *For M connected and simply connected, and G a finite group acting on M then there is an isomorphism of graded rings*

$$HH^*(C^*M\#G, C^*M\#G) \cong H_*^{\mathrm{pro}}\left(\mathcal{L}(M\times_G EG)^{-T(M\times_G EG)}\right)$$

between the Hochschild cohomology of $C^*M\#G$ and the homology of the free loop space pro ring spectrum associated to the orbifold $[M/G]$.

Proof. Consider the following sequence of graded ring isomorphisms

$$\begin{aligned}
&H_*^{pro}(\mathcal{L}(M\times_G EG)^{-T(M\times_G EG)})\\
&:= \lim_{\leftarrow n} H_*((P_G M\times_G EG_n)^{-e_0^*T(M\times_G EG_n)}) && (6.4)\\
&\cong \lim_{\leftarrow n} H_*\left(C_*\left((P_G M\times EG_n)^{-e_0^*T(M\times EG_n)}\right)^G\right) && (6.5)\\
&\cong \lim_{\leftarrow n} H_*\left(C_*\left(P_G M^{-TM}\wedge EG_n^{-TEG_n}\right)^G\right) && (6.6)\\
&\cong \lim_{\leftarrow n} H_*\left(\left(C_*(EG_n^{-TEG_n})\otimes C_*(P_G M^{-TM})\right)^G\right) && (6.7)\\
&\cong \lim_{\leftarrow n} H^*\left(\left(C^*(EG_n)\otimes C_*(P_G M^{-TM})\right)^G\right) && (6.8)\\
&\cong H^*\left(\lim_{\leftarrow n}\left(C^*(EG_n)\otimes C_*(P_G M^{-TM})\right)^G\right) && (6.9)\\
&\cong H^*\left(\left(C^*EG\otimes C_*(P_G M^{-TM})\right)^G\right) && (6.10)\\
&\cong \mathrm{Ext}^*_{\mathbb{Z}G}\left(\mathbb{Z}, C_*(P_G M^{-TM})\right) && (6.11)\\
&\cong HH^*(C^*M\#G, C^*M\#G)
\end{aligned}$$

where the isomorphism between (6.4) and (6.5) is due to the fact that G acts freely on EG_n; the isomorphism between (6.5) and (6.6) is due to the fact that as ring spectra $X^{-TX}\wedge Y^{-TY}$ and $(X\times Y)^{-T(X\times Y)}$ are homotopic; the isomorphism between (6.6) and (6.7) follows by the Eilenberg-Zilber theorem; the isomorphism between (6.7) and (6.8) follows by S-duality between $C_*(EG_n^{-TEG_n})$ and $C^*(EG_n)$; the isomorphisms between (6.8) and (6.9) follows from the fact that the inverse system of graded rings

$$H^*\left(\left(C^*EG_n\otimes C_*(P_G M^{-TM})\right)^G\right)$$

satisfies the Mittag-Leffler condition; the isomorphism between (6.9) and (6.10) follows from the fact that the inverse system $H^*(EG_n)$ of graded rings satisfies the Mittag-Leffler condition; the isomorphism between (6.10) and (6.11) follows from (6.2); and the last isomorphism is proved in theorem 5.9.

Then, the theorem follows from the previous isomorphisms. □

With theorem 6.3 in hand, we define

Definition 6.4. The string topology ring associated to a global quotient orbifold $[M/G]$ is the graded ring

$$H_*^{pro}(\mathcal{L}(M \times_G EG)^{-T(M \times_G EG)}; \mathbb{Z}).$$

Note that the previous definition is indeed a homotopy invariant of the orbifold, and hence, it is well defined in the Morita equivalence class of the orbifold $[M/G]$. The isomorphism with the Hochschild cohomology ring of $C^*M\#G$ depended explicitly on the fact that $H^1M = 0$, and hence it is needed that M be simply connected.

7. Applications

7.1. **Rational coefficients.** When we use rational coefficients, the G-invariants functor is exact, and therefore there is no need in deriving it. Hence the string topology ring of the orbifold $[M/G]$ becomes isomorphic as graded rings to

$$H_*^{pro}\left(\mathcal{L}(M \times_G EG)^{-T(M \times_G EG)}; \mathbb{Q}\right) \cong H_*(P_G M^{-TM}; \mathbb{Q})^G$$

which is precisely what was defined in [18] as the orbifold string topology ring.

So, we have that in the case that M is connected and simply connected, the orbifold string topology of $[M/G]$ with coefficients in $\mathbb{Q}$ is isomorphic to the ring

$$H_*(P_G M^{-TM}; \mathbb{Q})^G \cong HH^*(C^*(M; \mathbb{Q})\#G, C^*(M; \mathbb{Q})\#G).$$

7.2. $M =$ **point.** The string topology for $[*/G]$ turns out to be a ring that can be reproduced with the pull-push formalism (se [27]). Let us see first that as a graded $\mathbb{Z}$-module, we have an isomorphism

$$H_*^{\text{pro}}(\mathcal{L}BG^{-TBG}) \cong \bigoplus_{(g)} H^*(BC(g))$$

where (g) runs over the conjugacy classes of elements in G and $C(g)$ denotes the centralizer of g in G

The groupoid $[P_G M/G]$ becomes simply $[G/G]$ where G acts by conjugation on G, thus we obtain

$$(G \times_G EG_n)^{-TBG_n} = \bigsqcup_{(g)} (EG_n/C(g))^{-\pi^* TBG_n} \cong \bigsqcup_{(g)} (BC(g)_n)^{-TBC(g)_n}$$

with $\pi : BC(g)_n := EG_n/C(g) \to EG_n/G = BG_n$; the second equality follows from the fact that π is a cover map and therefore $TBC(g)_n \cong \pi^* TBG_n$.

Hence

$$\begin{aligned} H_*^{\mathrm{pro}}(\mathcal{L}BG^{-TBG}) &= \bigoplus_{(g)} \lim_{\leftarrow n} H_*\left((BC(g)_n)^{-TBC(g)_n}\right) \\ &= \bigoplus_{(g)} \lim_{\leftarrow n} H^*(BC(g)_n) \\ &= \bigoplus_{(g)} H^*(BC(g)). \end{aligned}$$

Now let us see what is the induced ring structure: we have the maps

$$EG_n/C(g) \times EG_n/C(h) \xleftarrow{\Delta} EG_n/(C(g) \cap C(h)) \longrightarrow EG_n/C(gh)$$

and all the pullbacks of the bundle TBG_n are isomorphic the corresponding tangent bundles. Note that the map Δ is injective and therefore we can perform the Thom-Pontrjagin construction giving us the map in homology

$$H_*(EG_n/C(g) \times EG_n/C(h)) \to H_{*-k_n}(EG_n/(C(g) \cap C(h)))$$

with $k_n = \dim(EG_n)$, which is Poincaré dual to the pull-back map in cohomology

$$\Delta^* : H^*(EG_n/C(g) \times EG_n/C(h)) \to H^*(EG_n/(C(g) \cap C(h))).$$

The natural map in homology

$$H_*(EG_n/(C(g) \cap C(h))) \to H_*(EG_n/C(gh))$$

is Poincaré dual to the push-foward map in cohomology

$$H^*(EG_n/(C(g) \cap C(h))) \to H^*(EG_n/C(gh))$$

that defines the induction map

$$H^*(B(C(g) \cap C(h))) \to H^*(BC(gh)).$$

We therefore see that the ring structure in $H_*^{\mathrm{pro}}(\mathcal{L}BG^{-TBG})$ is obtained by taking classes in $H^*BC(g)$ and $H^*BC(h)$ respectively, pulling them back to $H^*B(C(g) \cap C(h))$ and then pushing them forward to $H^*BC(gh)$. This procedure is what is known as the pull-push formalism, and it is a well known fact among algebraists that the ring structure $HH^*(\mathbb{Z}G, \mathbb{Z}G)$ could be recovered with this formalism (see Example 2.7 of [27] an the references therein).

Note that in the case that G is abelian we have

$$HH^*(\mathbb{Z}G, \mathbb{Z}G) \cong \mathbb{Z}G \otimes_{\mathbb{Z}} \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}G \otimes_{\mathbb{Z}} H^*BG$$

and therefore the string topology ring for $[*/G]$ is the ring $\mathbb{Z}G \otimes_{\mathbb{Z}} H^*BG$.

7.3. **Manifold with finite fundamental group.** Consider a connected compact manifold N with finite fundamental group $G = \pi_1 N$. Then by theorem 6.3

Theorem 7.1. *The string topology ring $H_*(\mathcal{L}N^{-TN};\mathbb{Z})$ as defined in [5] is isomorphic to the ring*

$$HH^*(C^*\tilde{N}\#G, C^*\tilde{N}\#G)$$

where $\tilde{N}$ is the universal cover of N.

Proof. Because G acts freely on $\tilde{N}$ we have that

$$H_*^{pro}\left(\mathcal{L}(\tilde{N}\times_G EG)^{-T(\tilde{N}\times_G EG)}\right) \cong H_*\left(P_G\tilde{N}^{-T\tilde{N}}\right)^G$$

and moreover we have that as ring spectra $P_G\tilde{N}^{-T\tilde{N}}/G \cong \mathcal{L}N^{-TN}$. Together with theorem 6.3 we have the desired isomorphism of graded rings

$$H_*(\mathcal{L}N^{-TN};\mathbb{Z}) \cong HH^*(C^*\tilde{N}\#G, C^*\tilde{N}\#G).$$

□

8. Failure of Hochschild cohomology invariance under orbifold equivalence

Taking a closer look at theorem 6.3, we see that we can only relate the string topology of an orbifold to the Hochschild cohomology ring of the group dg-ring, whenever we can write the orbifold as the groupoid $[M/G]$ with M simply connected and connected. The reason for this restriction lies in the use of the Eilenberg-Moore spectral sequence to relate the complex

$$C^*(M)\overset{L}{\otimes}_{C^*(M)^e} C^*(M) \quad \text{with} \quad C^*(\mathcal{L}M),$$

which in the case that M is not simply connected, it does not converge (see [6]).

But it is natural to ask to whether the Hochschild cohomology ring

$$HH^*(C^*M\#G, C^*M\#G)$$

independent of the presentation of the orbifold; in other words, for two Morita equivalent orbifold groupoids $[M/G]$ and $[N/H]$ in the sense of [20], are the graded rings

$$HH^*(C^*M\#G, C^*M\#G) \quad \text{and} \quad HH^*(C^*N\#H, C^*N\#H)$$

isomorphic?

The following result tells us that in general the answer of the previous question is negative.

Proposition 8.1. *Let $M = S^1$ be the circle equipped with the antipodal action of $G = \mathbb{Z}/2$. Let us replace the cochains by differential forms ΩM and let us work complex coefficients. Consider the Morita equivalent groupoids*

$[M/G]$ and the topological quotient M/G, together with the associated dg-rings $\Omega M\#G$ and $\Omega(M/G)$. Then $HH^*(\Omega M\#G, \Omega M\#G)$ is not isomorphic to $HH^*(\Omega(M/G), \Omega(M/G))$ as rings.

Proof. Let $\mathbb{C}[x]/x^2$ be the dg-algebra with trivial differential and $\deg x = 1$. Consider the trivial G-action on $\mathbb{C}[x]/x^2$. Note that $H^*(\Omega M)$ is spanned by the classes of 1 and $d\Theta$. Since the elements 1 and $d\Theta$ are G-invariant we get a G-equivariant quasi-isomorphism

$$\mathbb{C}[x]/x^2 \to \Omega M,\ x \mapsto d\Theta \tag{8.1}$$

Since M/G is diffeomorphic to M a straightforward computation now gives

$$HH^*(\Omega(M/G)) \cong HH^*(\Omega M) \cong HH^*(\mathbb{C}[x]/x^2) \cong \mathbb{C}[y,z]/z^2$$

where $\deg y = 0$ and $\deg z = 1$. On the other hand, the map in (8.1) gives a quasi-isomorphism

$$\mathbb{C}[x]/x^2\#G \cong \Omega M\#G.$$

Since $\mathbb{C}[x]/x^2\#G$ is the usual algebra tensor product of $\mathbb{C}[x]/x^2$ and $\mathbb{C}G$ and since $\mathbb{C}G$ is commutative and semi-simple this gives $HH^*(\Omega M\#G) \cong$

$$HH^*(\mathbb{C}[x]/x^2\#G) = HH^*(\mathbb{C}[x]/x^2) \otimes \mathbb{C}G \cong \mathbb{C}[y,z]/z^2 \otimes \mathbb{C}G.$$

This proves the proposition since $\mathbb{C}[y,z]/z^2$ and $\mathbb{C}[y,z]/z^2 \otimes \mathbb{C}G$ are non-isomorphic rings. □

Since Hochschild cohomology commutes with extension of scalars and ΩM is Morita equivalent to $C^*M \otimes_{\mathbb{Z}} \mathbb{C}$, we conclude that $HH^*(C^*M\#G, C^*M\#G)$ and $HH^*(C^*(M/G), C^*(M/G))$ are also non-isomorphic rings.

Because the Hochschild cohomology is invariant under derived equivalence (see [16]), we can conclude that

Corollary 8.2. *Let $M = S^1$ and endow it with the antipodal action of $G = \mathbb{Z}/2$. Then the derived categories of dg-modules*

$$\mathcal{D}(C^*M\#G) \quad \text{and} \quad \mathcal{D}(C^*(M/G))$$

(as in [24]) are not equivalent.

(Notice that by proposition 8.1 these categories are non-equivalent also with coefficients in the field of complex numbers.) Therefore one cannot expect that there is an isomorphism of Hochschild cohomology rings for the dg-rings of Morita equivalent groupoids, and in some sense, one can only produce an isomorphism with the string topology ring whenever one can find a description of the orbifold given by the quotient of a simply connected manifold by the action of a finite group. When this is not the case, as for example the case of manifolds with non-finite fundamental group, we do not know how to recover the string topology ring via the Hochschild cohomology of some dg-ring associated to the orbifold. We leave this question open.

9. Appendix A

In this section we will explain how the sign conventions arise for the Bar construction of section 2.2.1. The idea consists in transporting the standard differentials of the complex $\oplus_k \mathcal{A}^{k+2}$ to the complex $\oplus_k \left(\mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A}\right)$ via some chosen isomorphisms.

Let us recall from (2.1) that there is an isomorphism $s : \mathcal{A} \to \mathcal{A}[1]$; $a \mapsto sa$ such that $dsa = -sda$. Define the isomorphism

$$\Phi_k : \mathcal{A}^{k+2} \to \mathcal{A} \otimes \mathcal{A}[1]^k \otimes \mathcal{A}$$

as the composition of the maps

$$\left(\mathrm{Id} \otimes s \otimes \mathrm{Id}^{\otimes k}\right) \circ \cdots \circ \left(\mathrm{Id}^{\otimes k-1} \otimes s \otimes \mathrm{Id}^{\otimes 2}\right) \circ \left(\mathrm{Id}^{\otimes k} \otimes s \otimes \mathrm{Id}\right).$$

Because s is an odd map we get

$$\left(\mathrm{Id}^{\otimes i} \otimes s \otimes \mathrm{Id}^{\otimes k-i+1}\right)(a_0|\dots|a_{k+1}) = \\ (-1)^{|a_0|+\cdots+|a_i|}(a_0|\dots|a_i|sa_{i+1}|a_{i+2}\dots|a_{k+1})$$

and therefore

$$\Phi_k(a_0|\dots|a_{k+1}) = (-1)^{\sum_{i=0}^{k-1}(k-i)|a_i|}(a_0|sa_1|\dots|sa_k|a_{k+1}).$$

The standard differential for the Bar resolution is defined as the sum $\delta^0 + \cdots + \delta^k$ with

$$\delta^j : \mathcal{A}^{k+2} \to \mathcal{A}^{k+1}$$
$$(a_0|\dots|a_{k+1}) \mapsto (-1)^j(a_0|\dots|a_{j-1}|a_ja_{j+1}|a_{j+2}|\dots|a_{k+1}).$$

We will transport these maps δ^j as maps $\mathcal{A}\otimes\mathcal{A}[1]^k\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}[1]^{k-1}\otimes\mathcal{A}$ to define the differential defined in (2.5).

We have that

$$\Phi_k(a_0|\dots|a_ja_{j+1}|\dots|a_{k+1}) = \\ (-1)^{\sum_{i=0}^{j}(k-i-1)|a_i|+\sum_{i=j+1}^{k-1}(k-i)|a_i|}(a_0|sa_1|\dots|s(a_ja_{j+1})|\dots|sa_k|a_{k+1})$$

which implies that the induced map becomes

$$\bar{\delta}^j : \mathcal{A}\otimes\mathcal{A}[1]^k\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}[1]^{k-1}\otimes\mathcal{A}$$
$$(a_0|sa_1|\dots|sa_k|a_{k+1}) \mapsto (-1)^{\varepsilon_j}(a_0|sa_1|\dots|s(a_ja_{j+1})|\dots|sa_k|a_{k+1})$$

which satisfies $\bar{\delta}^j \circ \Phi_k = \Phi_{k-1} \circ \delta^j$ and with

$$\varepsilon_j := |a_0| + |a_1| + \cdots + |a_j| - j.$$

Note that when $j = k$ we get that

$$\bar{\delta}^k(a_0|sa_1|\dots|sa_k|a_{k+1}) = -(-1)^{\varepsilon_k}(a_0|sa_1|\dots|sa_{k-1}|a_ka_{k+1})$$

and therefore the differential δ in the Bar construction $B(\mathcal{A})$ becomes

$$\delta := \bar{\delta}^0 + \cdots + \bar{\delta}^k$$

which is the one defined in (2.5).

To finalize let us explain how the differential defined in (2.4) comes from the internal differential of $\mathcal{A}\otimes\mathcal{A}^k\otimes\mathcal{A}$ and the graded commutation of s and d:

$$\begin{aligned}
&d(a_0|sa_1|\dots|sa_k|a_{k+1})\\
&\quad=(da_0|sa_1|\dots|sa_k|a_{k+1})+\sum_{j=1}^{k}(-1)^{\varepsilon_{j-1}}(a_0|sa_1|\dots|dsa_j|\dots|sa_k|a_{k+1})\\
&\qquad+(-1)^{\varepsilon_k}(a_0|sa_1|\dots|sa_k|da_{k+1})\\
&\quad=(da_0|sa_1|\dots|sa_k|a_{k+1})-\sum_{j=1}^{k}(-1)^{\varepsilon_{j-1}}(a_0|sa_1|\dots|sda_j|\dots|sa_k|a_{k+1})\\
&\qquad+(-1)^{\varepsilon_k}(a_0|sa_1|\dots|sa_k|da_{k+1}).
\end{aligned}$$

10. Appendix B

Let M be a compact, oriented, differentiable, connected and simply connected manifold. In this section we resolve the technical problems about the Hochschild cohomology of the singular cochains $C^* := C^*(M)$ that arise from the fact that C^* is not a free $\mathbb{Z}$-module (unless M is a point). This is so because an infinite product of copies of $\mathbb{Z}$ is not free over $\mathbb{Z}$. This problem would disappear if one uses coefficients over a field, but we prefer to develop the theory over $\mathbb{Z}$ as it will turn out that with some effort it is possible to do so. On the algebraic side we could solve this problem by considering simplicial cochains S^* instead of singular cochains (having an explicit simplicial decomposition of M). Note however that the algebraic object that naturally corresponds to the topological side of this paper is the Hochschild cohomology of C^* and not that of S^*, so we are forced to deal with C^*.

Recall that one has

$$HH^*(C^*,C^*)=H^*\mathcal{H}om_{C^{*e}}(B(C^*),C^*)$$

and that

$$HH^*(S^*,S^*)=H^*\mathcal{H}om_{S^{*e}}(B(S^*),S^*)\cong \operatorname{Ext}_{S^{*e}}(S^*,S^*).$$

The fact that $C^*(M)$ is not free over $\mathbb{Z}$ implies that $B(C^*)$ cannot be cofibrant in the sense of section 2. Nevertheless, this quantity can be interpreted as a *relative* Ext-group (see lemma 9.1.3 in [26]) and it is even possible to define a model category structure on the category of dg-modules over C^{*e} in which $B(C^*)$ is cofibrant. In principle, all the homological algebra of this paper could be translated to this setup, but it turns out that this is not necessary since we shall prove that

Theorem 10.1. *There is a canonical isomorphism*

$$HH^*(C^*,C^*)\cong HH^*(S^*,S^*)$$

as graded $\mathbb{Z}$-algebras.

The reason for this to be possible is that C^* is the dual of C_* which is free over $\mathbb{Z}$. We have

Lemma 10.2. *Let (X, d) and (Y, ∂) be (unbounded) complexes of free $\mathbb{Z}$-modules such that $H^*(X) \cong H^*(Y)$. Then X and Y are homotopic and any quasi-isomorphism $q : X \to Y$ is a homotopy equivalence.*

Proof. For every n we have that $X^n \to \operatorname{Im} d^n$ splits, since $\operatorname{Im} d^n$ is a free $\mathbb{Z}$-module, being a submodule of the free $\mathbb{Z}$-module X^{n+1}. Therefore $X^n \cong \operatorname{Ker} d^n \oplus \operatorname{Im} d^n$ and the differential $d^n : X^n \to X^{n+1}$ corresponds to $(a, b) \mapsto (b, 0)$.

Let A_n denote the subcomplex $\operatorname{Im} d^{n-1} \hookrightarrow \operatorname{Ker} d^n$ of X. It follows that

$$X \cong \oplus_{n\in\mathbb{Z}} A_n$$

Similarly, $Y \cong \oplus_{n\in\mathbb{Z}} B_n$, where B_n is the subcomplex $\operatorname{Im} \partial^n \hookrightarrow \operatorname{Ker} \partial^{n+1}$ of Y.

Since $H^n(A_n) \cong H^n(X) \cong H^n(Y) \cong H^n(B_n)$ and A_n is a projective resolution of $H^n(A_n)[-n]$ and B_n is a projective resolution of $H^n(B_n)[-n]$ we conclude that A_n is homotopic to B_n. Thus X is homotopic to Y.

For the last assertion, note that each A_n and B_n are cofibrant objects of $C(\mathbb{Z})$. Thus X and Y are cofibrant as well and hence any quasi-isomorphism between them is a homotopy. □

We fix a finite simplicial decomposition of M on which G acts simplicialy and we consider the simplicial decomposition that the barycentric subdivision defines (this is in order to get a simplicial decomposition which induces a G-CW decomposition). If we think of a k-simplex in M as a certain map from Δ_k to M it is clear how to get an embedding $i_* : S_* \to C_*$ which is a G-equivariant quasi-isomorphism. Moreover, if we denote by $f = i^* : C^* \to S^*$ the corresponding map on cochains, then f is a morphism of dg-algebras. We have

Proposition 10.3. *$f : C^* \to S^*$ is a homotopy equivalence.*

Proof. Since S_* and C_* are free $\mathbb{Z}$-modules, lemma 10.2 shows that i_* is a homotopy-equivalence. Hence also $f = i^*$ is a homotopy equivalence. □

We can now prove

Proof of theorem 10.1. The dg-ring map $f : C^* \to S^*$ induces a C^{*e}-module structure on S^* and a $B(C^*)^e$-linear map

$$\overline{f} : B(C^*) \to B(S^*),\ (c_0| \dots |c_{k+1}) \mapsto (f(c_0)| \dots |f(c_{k+1})).$$

Note that f will not have a multiplicative homotopy inverse and hence that $\overline{f}$ is not a homotopy equivalence.

Nevertheless, we have canonical maps

$$\mathcal{H}om_{C^{*e}}(B(C^*), C^*) \to \mathcal{H}om_{C^{e*}}(B(C^*), S^*),\ \phi \mapsto f \circ \phi$$
$$\mathcal{H}om_{S^{*e}}(B(S^*), S^*) \to \mathcal{H}om_{C^{*e}}(B(C^*), S^*),\ \phi \mapsto \phi \circ \overline{f}$$

that we claim are both quasi-isomorphisms. This would prove theorem 10.1.

To show that each of the previous maps are q.i.'s, we will use specific spectral sequences on both sides of each map, whose zeroth differentials will avoid the part of the total differential that reflects the multiplication, and that will therefore become isomorphic at the first page. Let us be more explicit.

For $\mathcal{A}$ a dg-ring define $T(\mathcal{A}) = \oplus_{i\geq 0}\mathcal{A}^{\otimes i}[i]$. Consider the restriction isomorphisms

$$\begin{aligned}\mathcal{H}om_{\mathcal{A}^e}(\mathcal{A}^{\otimes i+2}[i],\mathcal{A}) \to& \mathcal{H}om_{\mathbb{Z}}(\mathcal{A}^{\otimes i}[i],\mathcal{A})\\ f \mapsto& \{\mathbf{a} \mapsto f(1\otimes \mathbf{a}\otimes 1)\}\end{aligned}$$

whose inverse isomorphism is

$$g \mapsto \{(a_0|\dots|a_{i+1}) \mapsto a_0 g(a_1|\dots|a_i)a_{i+1}\}.$$

If we use these isomorphisms to transport the differential in $\mathcal{H}om_{\mathcal{A}^e}(B(\mathcal{A}),\mathcal{A})$ to $\mathcal{H}om_{\mathbb{Z}}(T(\mathcal{A}),\mathcal{A})$ we get an isomorphisms of these two homomorphism complexes. Explicitly, the differential d on $\mathcal{H}om_{\mathbb{Z}}(T(\mathcal{A}),\mathcal{A})$ is given by

$$df(\mathbf{a}) = d_t(f(\mathbf{a})) - (-1)^{|f|}(\delta f(\mathbf{a}) + f(d_s(\mathbf{a})))$$

where d_t is the differential of $\mathcal{A}$, the target of the homomorphism, d_s is the differential on the source, which becomes

$$d_s(a_1|...|a_k) = -\sum_{i=1}^{k}(-1)^{\varepsilon_{i-1}}(a_1|...|da_i|...|a_k),$$

and δf is the homomorphism defined as

$$\begin{aligned}\delta f(a_1|...|a_k) =& a_1 f(a_2|...|a_k) + \sum_{i=2}^{k-1}(-1)^{\varepsilon_i} f(a_1|...|a_i a_{i+1}|...|a_k)\\ &- (-1)^{\varepsilon_k} f(a_1|...|a_{k-1})a_k.\end{aligned}$$

Applying the previous discussion to $\mathcal{A} = C^*$ and $\mathcal{A} = S^*$ we see that it suffices to show that the following two dg-maps are quasi-isomorphisms:

$$\mathcal{H}om_{\mathbb{Z}}(T(C^*),C^*) \to \mathcal{H}om_{\mathbb{Z}}(T(C^*),S^*),\ \phi \mapsto f\circ\phi$$

$$\mathcal{H}om_{\mathbb{Z}}(T(S^*),S^*) \to \mathcal{H}om_{\mathbb{Z}}(T(C^*),S^*),\ \phi \mapsto \phi\circ\overline{f}$$

Let us prove that the first map is a quasi-isomorphism

Lemma 10.4. *The map*

$$\Phi : \mathcal{H}om_{\mathbb{Z}}(T(C^*),C^*) \to \mathcal{H}om_{\mathbb{Z}}(T(C^*),S^*),\ \phi \mapsto f\circ\phi$$

is a quasi-isomorphism.

Proof. Consider the filtrations

$$F_k = \mathcal{H}om_{\mathbb{Z}}\left(\bigoplus_{i=k}^{\infty} C^{*\otimes i}[i], C^*\right) \qquad \overline{F}_k = \mathcal{H}om_{\mathbb{Z}}\left(\bigoplus_{i=k}^{\infty} C^{*\otimes i}[i], S^*\right)$$

of $\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*)$ and $\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*)$ respectively, which are compatible with the homomorphism Φ.

The associated spectral sequences associated to F_* and $\overline{F}_*$ have for zeroth page the complexes

$$E_0 = \bigoplus_{k=0}^{\infty} F_k/F_{k+1} = \bigoplus_{k=0}^{\infty} \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k], C^*)$$
$$\overline{E}_0 = \bigoplus_{k=0}^{\infty} \overline{F}_k/\overline{F}_{k+1} = \bigoplus_{k=0}^{\infty} \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k], S^*)$$

and whose zeroth differentials are

$$d^0\phi(\mathbf{a}) = d_{C^*}(\phi(\mathbf{a})) - (-1)^{|\phi|}\phi(d_s(\mathbf{a}))$$
$$\overline{d}^0\overline{\phi}(\mathbf{a}) = d_{S^*}(\overline{\phi}(\mathbf{a})) - (-1)^{|\overline{\phi}|}\overline{\phi}(d_s(\mathbf{a})).$$

where d_s is the internal differential of the complex $C^{*\otimes k}[k]$.

Applying the canonical isomorphisms

$$\mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k], C^*) \cong \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k] \otimes C_*, \mathbb{Z})$$
$$\mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k], S^*) \cong \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}[k] \otimes S_*, \mathbb{Z})$$

we see that the induced differentials on the right hand side, are just the canonical differentials given by the dualization of the tensor product of d_s with ∂_{C_*} and of d_s and ∂_{S_*} respectively, where ∂ denotes the differential at the chains level.

Because the map $i : S_* \to C_*$ induces a homotopy equivalence, we see that the induced map on the zeroth pages

$$\Phi : E_0 \to \overline{E}_0$$

is a quasi-isomorphism, and therefore the induced homomorphism on the first pages becomes an isomorphism

$$\Phi : E_1 \xrightarrow{\cong} \overline{E}_1.$$

We have therefore that the spectral sequences are isomorphic after the first page and moreover that the filtrations are both Hausdorff and weakly convergent.

The convergence of the spectral sequences tell us that Φ induces an isomorphism between the inverse limits (see Theorem 3.9 of [19])

$$\begin{aligned} &\Phi : \lim_{\leftarrow k} \left(H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*)/\mathrm{Im}(H^*F_k \to H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*))\right) \\ &\xrightarrow{\cong} \lim_{\leftarrow k} \left(H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*)/\mathrm{Im}(H^*\overline{F}_k \to H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*)\ , S^*))\right) \end{aligned} \tag{10.1}$$

but as the cohomology groups $H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*)$ and $H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*)$ are both graded groups, and in each degree they are finitely generated (as we know that $H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*)$ is isomorphic to the homology of $\mathcal{L}M^{-TM}$),

then we have that (10.1) implies that there is Φ induce an isomorphism at the level of the cohomologies

$$\Phi : H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), C^*) \overset{\cong}{\to} H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*).$$

This finishes the proof of lemma 10.4. □

Let us prove that the second map is a quasi-isomorphism

Lemma 10.5. *The map*

$$F : \mathcal{H}om_{\mathbb{Z}}(T(S^*), S^*) \to \mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*),\ \phi \mapsto \phi \circ \overline{f}$$

is a quasi-isomorphism.

Proof. Consider the following double filtrations

$$P^{j.k} = \mathcal{H}om_{\mathbb{Z}}(\bigoplus_{i \geq k} S^{*\otimes i}, S^{\geq j})$$
$$\overline{P}^{j.k} = \mathcal{H}om_{\mathbb{Z}}(\bigoplus_{i \geq k} C^{*\otimes i}, S^{\geq j})$$

and note that the filtrations are compatible with the map F, and that the differentials δ and $d_t = d_{S^*}$ raise the degree, namely we have that

$$\delta : P^{j,k} \to P^{j,k+1}, d_t : P^{j,k} \to P^{j+1,k},\quad \delta : \overline{P}^{j,k} \to \overline{P}^{j,k+1}$$
$$\text{and } d_t : \overline{P}^{j,k} \to \overline{P}^{j+1,k}.$$

Let us take the filtrations defined by these double filtrations

$$Q^r = \sum_{k+j=r} P^{j,k} \qquad \overline{Q}^r = \sum_{k+j=r} \overline{P}^{j,k}$$

noting that both δ and d_t raise the degree $\delta, d_t : Q^r \to Q^{r+1}$. Therefore the associated spectral sequences E_* and $\overline{E}_*$ are compatible via the map F

$$F : E_* \to \overline{E}_*$$

and on the zeroth pages of both spectral sequences we get the associated graded

$$E_0 \cong \bigoplus_{r=0}^{\infty} \bigoplus_{k+j=r} \mathcal{H}om_{\mathbb{Z}}(S^{*\otimes k}, S^j)$$

$$\overline{E}_0 \cong \bigoplus_{r=0}^{\infty} \bigoplus_{k+j=r} \mathcal{H}om_{\mathbb{Z}}(C^{*\otimes k}, S^j).$$

The zeroth differential d^0 on the group $\mathcal{H}om_{\mathbb{Z}}(S^{*\otimes k}, S^j)$ becomes the differential obtained by pre-composing with the internal differential of the source $S^{*\otimes k}$

$$(d^0\phi)(a_1|...|a_k) = (-1)^{|\phi|}\phi(d(a_0|...|a_k))$$

(the same happens with the zeroth differential $\overline{d}^0$ of $\overline{E}_0$), and therefore we have that the first level of the spectral sequences become the sum of the cohomologies of the duals of $S^{\otimes k}$ and $C^{\otimes k}$ respectively tensored with S^*

$$E_1 \cong \bigoplus_{r=0}^{\infty} H^*((S^{*\otimes r})^\vee) \otimes S^*$$
$$\overline{E}_1 \cong \bigoplus_{r=0}^{\infty} H^*((C^{*\otimes r})^\vee) \otimes S^*$$

as we know that S^* is a finitely generated free $\mathbb{Z}$-module.

The map $F : E_1 \to \overline{E}_1$ at the first level is clearly an isomorphism as the map $f : C^* \to S^*$ induces a quasi-isomorphism at the level of the tensor products $C^{*\otimes k} \xrightarrow{\simeq} S^{*\otimes k}$ and their duals $(S^{*\otimes k})^\vee \xrightarrow{\simeq} (C^{*\otimes k})^\vee$.

We have now that F becomes an isomorphism at the first level of the spectral sequences. We also have that the filtration given by Q^r is Hausdorff $\cap_r Q^r = \{0\}$, therefore weakly-convergent, and applying the same argument as in the previous lemma, namely that the cohomologies

$$H^*\mathcal{H}om_{\mathbb{Z}}(T(S^*), S^*) \quad \text{and} \quad H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*)$$

are both graded rings, we have that the isomorphism of the spectral sequences implies that the cohomologies

$$H^*\mathcal{H}om_{\mathbb{Z}}(T(S^*), S^*) \quad \text{and} \quad H^*\mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*)$$

are isomorphic. Therefore F induces a q.i. on the complexes

$$\mathcal{H}om_{\mathbb{Z}}(T(S^*), S^*) \xrightarrow{\simeq} \mathcal{H}om_{\mathbb{Z}}(T(C^*), S^*).$$

□

This ends the proof of Theorem 10.1. □

Using the notation of (3.6) we see that theorem 10.1 implies that there are canonical isomorphisms

(10.2)
$$H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*_g) \cong H^*\mathcal{H}om_{S^{*e}}(B(S^*), S^*_g) = \mathrm{Ext}_{S^{*e}}(S^*, S^*_g)$$
$$H^*\mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G) \cong H^*\mathcal{H}om_{S^{*e}}(B(S^*), S^*\#G) = \mathrm{Ext}_{S^{*e}}(S^*, S^*\#G)$$

which induces a canonical isomorphism between the cohomologies

$$H^*\mathcal{H}om_{\mathbb{Z}G}(\overline{B}(\mathbb{Z}G), \mathcal{H}om_{C^{*e}}(B(C^*), C^*\#G)) \cong$$
$$H^*\mathcal{H}om_{\mathbb{Z}G}(\overline{B}(\mathbb{Z}G), \mathcal{H}om_{S^{*e}}(B(S^*), S^*\#G)).$$

Therefore we can conclude that

Corollary 10.6. *There is a canonical ring isomorphism*

$$HH^*(C^*\#G, C^*\#G) \cong HH^*(S^*\#G, S^*\#G)$$

which can also be seen as a ring isomorphism

$$HH^*(C^*\#G, C^*\#G) \cong \mathrm{Ext}_{S^*\#G^e}(S^*\#G, S^*\#G).$$

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Hausdorff Center for Mathematics, Landwirtschaftskammer (Neubau), Endenicher Allee 60, D-53115 Bonn, GERMANY

E-mail address: `andresangelalumni@googlemail.com`

Departamento de Matemáticas, Universidad de los Andes, Carrera 1 N. 18A - 10, Bogotá, COLOMBIA

E-mail address: `erbackel@uniandes.edu.co`

Mathematisches Institut, Westfälische Wilhelms-Universität, Einsteinstrasse 62, 48149 Münster, GERMANY

Departamento de Matemáticas, Universidad de los Andes, Carrera 1 N. 18A - 10, Bogotá, COLOMBIA

E-mail address: burib_01@uni-muenster.de, buribe@uniandes.edu.co